Reflective centers of module categories

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SUMMARY

- Given $\mathcal{C}$ braided monoidal category, $\mathcal{M}$ a $\mathcal{C}$-module category
  - Construct a *braided* module category $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$ called the reflective center
- Assume $\mathcal{C} = H$–**mod**, H a quasi-triangular Hopf algebra, $\mathcal{M} = A$–**mod** for a left H-comodule algebra A
  - Then $\mathcal{E}_{\mathcal{C}}(\mathcal{M}) \simeq R_H(A)$–**mod**, where $R_H(A)$ is the reflective algebra associated to A
  - $R_H(A)$ is a smash product of A and Majid's braided group $\widehat{H}^*$ with covariantized product
  - The algebra $\widehat{H}^*$ is also called the *reflection equation algebra*

Joint work with Chelsea Walton (Rice University) & Milen Yakimov (Northeastern University) `ArXiv:2307.14764`

CONTENTS

MOTIVATION

The following table appeared in a presentation of Martina Balagovic (on her paper with Stefan Kolb [BK19]):

If you like:	... then you should also like:
1. Quantum enveloping algebras	1. Quantum symmetric pairs
2. Universal quantum *R*-matrices	2. Universal quantum *K*-matrices
3. The quantum Yang–Baxter equation	3. The quantum reflection equation
4. Braided tensor categories	4. Braided module categories

The following additional lines summarize our recent work:

Classical constructions and notions:	Our constructions and notions:
5. Drinfeld centers of tensor categories	5. *Reflective centers* of module categories
6. Yetter–Drinfeld modules	6. Doi–Hopf modules
7. Drinfeld doubles of Hopf algebras	7. *Reflective algebras* of comodule algebras

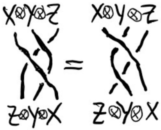
BRAIDED MONOIDAL CATEGORIES

A braided (strict) monoidal category $\mathcal{C}$ has:

- a *tensor product* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- a *unit* $\mathbb{1} \in \mathcal{C}$
- a *braiding* $c_{X,Y} =$  $: X \otimes Y \xrightarrow{\sim} Y \otimes X$ satisfying:

$c_{X,Y\otimes Z} =$  $=$  $= (\mathrm{Id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{Id}_Z)$

$c_{X\otimes Y,Z} =$  $=$  $= (c_{X,Z} \otimes \mathrm{Id}_Y)(\mathrm{Id}_X \otimes c_{Y,Z})$

$\Rightarrow$ 

MODULE CATEGORIES

Classical concept:	Categorical analogue:
Ring R	Monoidal category $\mathcal{C}$
Commutative ring	Braided monoidal category
Center $Z(R)$	Drinfeld center $\mathcal{Z}(\mathcal{C})$
R-module M	$\mathcal{C}$-module category $\mathcal{M}$

Definition (Module category)

A $\mathcal{C}$-module category $\mathcal{M}$ is a category $\mathcal{M}$ with action functor

$$\triangleright \colon \mathcal{C} \times \mathcal{M} \to \mathcal{M}$$

with coherent isomorphisms: $u_M \colon \mathbb{1} \triangleright M \xrightarrow{\sim} M$ and

$$m_{X,Y,M} \colon (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M)$$

Simplification: $\mathcal{M}$ is *strict*, i.e., $m_{X,Y,M} = \mathrm{Id}$ and $u_M = \mathrm{Id}$.

BRAIDED MODULE CATEGORIES

Fix $\mathcal{C}$ — braided monoidal category, with braiding

$$c_{X,Y} \colon X \otimes Y \xrightarrow{\sim} Y \otimes X$$

Definition

A left $\mathcal{C}$-module category $\mathcal{M}$ with action functor $\triangleright \colon \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is *braided* if it is equipped with a natural isomorphism

$$e_{X,M} =\colon X \triangleright M \xrightarrow{\sim} X \triangleright M$$

called the *braiding*, satisfying (strict case)

$$\begin{aligned} e_{X\otimes Y,M} &= (\mathrm{Id}_X \triangleright e_{Y,M}) \circ (c_{Y,X} \triangleright \mathrm{Id}_M) \circ (\mathrm{Id}_Y \triangleright e_{X,M}) \circ (c^{-1}_{Y,X} \triangleright \mathrm{Id}_M), \\ e_{X,Y\triangleright M} &= (c_{Y,X} \triangleright \mathrm{Id}_M) \circ (\mathrm{Id}_Y \triangleright e_{X,M}) \circ (c_{X,Y} \triangleright \mathrm{Id}_M) \end{aligned}$$

- Braided module categories were defined by Brochier [Bro13]
- General theory developed by Kolb [Kol20] and others

BRAIDED MODULE CATEGORIES

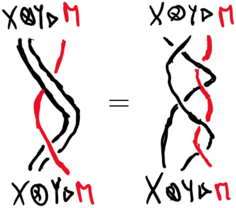
The axioms can be visualized using graphical calculus:

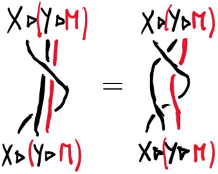
$c_{X,Y} =$ 

$: X \otimes Y \xrightarrow{\sim} Y \otimes X$

$e_{X,M} =$ 

$: X \triangleright M \xrightarrow{\sim} X \triangleright M$





COMODULE ALGEBRAS

- $\mathcal{C} = H$–**mod** — H a quasitriangular Hopf algebra
- Given A an H-comodule algebra with H-coaction

$\delta =$  $: A \to H \otimes A,$

- construct a $\mathcal{C}$-module category structure on A–**mod**

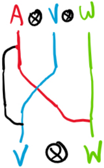
$$\triangleright : H\text{–}\mathbf{mod} \times A\text{–}\mathbf{mod} \to A\text{–}\mathbf{mod},$$

$\Big((V, a_V =$  $), (W, a_W =$  $)\Big) \mapsto V \triangleright W = (V \otimes W, a_{V\otimes W}),$

- with action given by

$$a_{V\otimes W} = (a_V \otimes a_W)(\mathrm{Id}_H \otimes \tau_{A,V} \otimes \mathrm{Id}_W)(\delta \otimes \mathrm{Id}_{V\otimes W})$$

- Graphically: $a_{V\otimes W} =$ 

$=$ 

QUASITRIANGULAR COMODULE ALGEBRAS

Question

When is $\mathcal{M} = A$–**mod** a braided module category?

The natural isomorphism $e_{V,W} \colon V \triangleright W \xrightarrow{\sim} V \triangleright W$ is given by the action of a *convolution invertible* element

$$K \in H \otimes A.$$

Answer

$\mathcal{M} = A$–**mod** *is a braided module category if and only if K is a quantum K-matrix, i.e., satisfies*

$$(\Delta \otimes \mathrm{Id}_A)K = K_{23}R_{21}K_{13}R_{21}^{-1} \qquad \text{in } H \otimes H \otimes A$$
$$(\mathrm{Id}_H \otimes \delta)K = R_{21}K_{13}R_{12} \qquad \text{in } H \otimes H \otimes A$$
$$\delta(a)K = K\delta(a), \qquad \forall a \in A \qquad \text{in } H \otimes A.$$

QUANTUM K-MATRICES

- Given a quasitriangular H-comodule algebra, the action of K on A-modules gives solution to the *quantum reflection equation*
- Most notable examples of quasitriangular H-comodule algebras are *coideal subalgebras* of quantum groups $U_q(\mathfrak{g})$
- These coideal subalgebras are based on quantum symmetric pairs and were constructed by G. Letzter [Let99]
- Quantum versions of symmetric pairs $\big(U(\mathfrak{k}), U(\mathfrak{g})\big)$, where $\mathfrak{k}$ are the fixed points of an involution $\theta\colon \mathfrak{g} \to \mathfrak{g}$
- Other authors have generalized Letzter's construction, e.g., Kolb–Yakimov [KY20] to Nichols algebras

DEFINITION OF THE REFLECTIVE CENTER

$\mathcal{C}$ braided monoidal category module category $\mathcal{M}$

Definition (L.–Walton–Yakimov)

The *reflective center* $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$ consists of

- objects which are pairs (M, e^M), where $M \in \mathcal{M}$ and e^M is a *half-braiding*

$$e^M_X =: X \triangleright M \xrightarrow{\sim} X \triangleright M,$$

  natural in Y satisfying

$$e^M_{X\otimes Y} \;=\; (\mathrm{Id}_X \triangleright e^M_Y) \circ (c_{Y,X} \triangleright \mathrm{Id}_M) \circ (\mathrm{Id}_Y \triangleright e^M_X) \circ (c^{-1}_{Y,X} \triangleright \mathrm{Id}_M),$$

- Morphisms $f\colon (M, e^M) \to (N, e^N)$ are $f \in \mathrm{Hom}_{\mathcal{M}}(M, N)$s.t:

$$\begin{array}{ccc} X \triangleright M & \xrightarrow{e^M_X} & X \triangleright M \\ \downarrow{\scriptstyle X\triangleright f} & & \downarrow{\scriptstyle X\triangleright f} \\ X \triangleright N & \xrightarrow{e^N_X} & X \triangleright N \end{array}$$

THE REFLECTIVE CENTER IS BRAIDED

Proposition (L.–Walton–Yakimov)

The reflective center $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$ is a braided $\mathcal{C}$-module category with

- *$\mathcal{C}$-action*

$$Y \triangleright (M, e^M) = (Y \triangleright M, e^{Y \triangleright M}), \quad \textit{where}$$

  - $e_X^{Y \triangleright M} \; = \; (c_{Y,X} \triangleright \mathrm{Id}_M) \circ (\mathrm{Id}_Y \triangleright e_X^M) \circ (c_{X,Y} \triangleright \mathrm{Id}_M)$

- *braiding given by*

$$e_{X,(M,e^M)} := e_X^M \colon X \triangleright M \xrightarrow{\sim} X \triangleright M.$$

PROPERTIES OF THE REFLECTIVE CENTER

- As a $\mathcal{C}$-module category, $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$ equals the $\mathcal{C}$-bimodule center

  $$\mathcal{Z}_{\mathcal{C}}(\mathcal{M}_{\mathrm{bim}}),$$

  where $\mathcal{M}_{\mathrm{bim}} = \mathcal{M}$ with a natural $\mathcal{C}$-bimodule structure obtained from $\mathcal{C}$ being braided
- Hence, $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$ is also a module category over the Drinfeld center $\mathcal{Z}(\mathcal{C})$
- $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$ is abelian when $\mathcal{M}$ is exact and finite
- $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$ is finite when $\mathcal{C}$ is finite and $\mathcal{M}$ is exact and finite
- $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$ is semisimple when $\mathcal{C}$ and $\mathcal{M}$ are finite and semisimple

HALF-BRAIDINGS GIVE COACTIONS

From now on assume that:

- $\mathcal{C} = H$–**mod**
- $\mathcal{M} = A$–**mod** for an *H*-comodule algebra A

Goal

A more concrete description of the reflective center $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$

Analogy: The Drinfeld center $\mathcal{Z}(H\text{–}\mathbf{mod})$ is equivalent to *H-Yetter–Drinfeld modules* (or *crossed modules*).

Ansatz: Given an object (M, e^M) in $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$ define

$$\delta_M := e^M_{H^{\text{reg}}}(\text{Id}_M \otimes 1_H) \colon M \longrightarrow H \otimes M$$

$\Longrightarrow$ This will give a *coaction*!

…of Majid's *covariantized coalgebra* $\widehat{H}$

THE COVARIANTIZED COALGEBRA

H — quasitriangular Hopf algebra

Define a new coproduct

$$\widehat{\Delta}\colon H \to H \otimes H, \qquad \widehat{\Delta}(h) := R_1^{(2)} h_{(1)} R_2^{(2)} \otimes h_{(2)} R_2^{(2)} S^{-1}(R^{(1)})$$

Proposition (Majid, 1991)

Denote by $\widehat{H}$ the algebra H with coproduct $\widehat{\Delta}$, the same counit, and left adjoint H-action

$$\ell \rightharpoonup h := \ell_{(2)} h S^{-1}(\ell_{(1)}),$$

*for all $h \in \widehat{H}$ and $\ell \in H$. Then $\widehat{H}$ is a **Hopf algebra** in H–**mod**.*

We will only use that $\widehat{H}$ is a left H-comodule algebra via

$$\widehat{\Delta}\colon \widehat{H} \to H \otimes \widehat{H}$$

COMODULES OVER $\widehat{H}$

Lemma

Given an object (M, e^M) of $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$, the map

$$\delta_M := e^M_{H^{\mathrm{reg}}}(\mathrm{Id}_M \otimes 1_H)$$

makes M a left $\widehat{H}$-comodule.

Recall that M is also a left A-module.

Question

What is the compatibility between $\widehat{H}$-coaction and A-action?

DOI–HOPF MODULES

For a Hopf algebra H consider:

- a left H-*comodule algebra* B, with coaction

$$\delta\colon B \to H \otimes B, \quad b \mapsto b^{[-1]} \otimes b^{[0]}$$

- a left H-*module coalgebra* C, with action

$$\rightharpoonup\colon H \otimes C \to C, \quad h \otimes c \mapsto h \rightharpoonup c$$

Definition (Doi, 1992)

Define the category ${}^{C}_{B}\mathbf{DH}(H)$ of C-B-Doi–Hopf modules whose

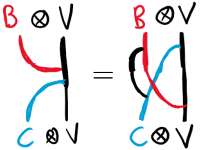
- **objects** M have the structure of a left B-*module* $b \otimes m \mapsto b \cdot m$ and C-*comodule* $c \mapsto m^{(-1)} \otimes m^{(0)}$ satisfying

$$(b \rightharpoonup m)^{(-1)} \otimes (b \cdot m)^{(0)} \;=\; (b^{[-1]} \cdot m^{(-1)}) \otimes (b^{[0]} \cdot m^{(0)})$$

- **morphisms** are *simultaneously* left B-module map and a left C-comodule maps

Doi–Hopf modules

The Doi–Hopf compatibility condition can be visualized as



Proposition (L.–Walton–Yakimov)

The reflective center $\mathcal{E}_{H\text{–}\mathbf{mod}}(A\text{–}\mathbf{mod})$ *is isomorphic to the category* ${}^{\widehat{H}}_{A}\mathbf{DH}(H)$ *of* $\widehat{H}$*-*A*-Doi–Hopf modules.*

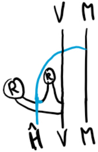
This makes the latter Doi–Hopf module category

- a *braided* $\mathcal{C}$-module category
- a $\mathcal{Z}(H\text{–}\mathbf{mod}) \cong {}^{H}_{H}\mathbf{YD}$-module category

DOI–HOPF MODULES

The **categorical action** $\triangleright$: H–**mod** $\times$ A–**mod** $\to$ A–**mod** extends to ${}^{\widehat{H}}_{A}\mathbf{DH}(H)$ with $\widehat{H}$-coaction

$$\delta^{V \triangleright M}(v \otimes m) = R_1^{(2)} m^{(-1)} R_2^{(1)} \otimes (R_2^{(1)} R_1^{(2)} \cdot v) m^{(0)} \quad \delta^{V \triangleright M} =$$



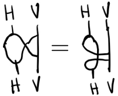
The *braiding* of a Doi–Hopf module M is:

$$e^M(v \otimes m) = (m^{(-1)} \cdot v) \otimes m^{(0)}, \quad e^M =$$

A YETTER–DRINFELD MODULE ACTION

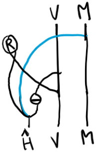
- Yetter–Drinfeld modules are H-modules and comodules s.t.

$$h_{(1)}v^{(-1)} \otimes (h_{(2)} \cdot v^{(0)}) = (h_{(1)} \cdot v)^{(-1)} h_{(2)} \otimes (h_{(1)} \cdot v)^{(0)})$$



- The H–**mod**-action *extends* to a categorical action by Yetter–Drinfeld modules ${}^{H}_{H}\mathbf{YD}$ on ${}^{\widehat{H}}_{A}\mathbf{DH}(H)$:
  - with the same A-action
  - $\widehat{H}$-coaction given by

$$\delta^{V \triangleright M}(v \otimes m) = R^{(2)} m^{(-1)} S^{-1}(v^{(-1)}) \otimes (R^{(1)} \cdot v^{(0)}) \otimes m^{(0)}, \quad \delta^{V \triangleright M} =$$



- The ${}^{H}_{H}\mathbf{YD}$-action restricts to the braided H–**mod** action along the braided monoidal functor $H\text{–}\mathbf{mod} \to {}^{H}_{H}\mathbf{YD}$

The reflective algebra

Definition (L.–W.–Y.)

The reflective algebra $R_H(A)$ of an H-comodule algebra A is the smash product algebra

$$R_H(A) = A \rtimes (\widehat{H}^*)^{\mathrm{op}},$$

with subalgebras A and $(\widehat{H}^*)^{\mathrm{op}}$

Lemma

There is an equivalence of categories between $\mathcal{E}_{H\text{–}\mathbf{mod}}(A\text{–}\mathbf{mod})$ *and* $R_H(A)$–**mod**.

This way:

- $R_H(A)$ becomes a *quasitriangular* H-comodule algebra
- the lemma upgrades to an equivalence of *braided module categories*

QUASITRIANGULAR COMODULE ALGEBRA STRUCTURE

1. The H-coaction δ_{ref} of $R_H(A)$ is given by
$$\delta(af) = \langle f_{(1)}, R^{(2)}\rangle a^{(-1)}R^{(1)} \otimes S^{-1}(f_{(3)}) \otimes a^{(0)}f^{(2)},$$
for all $a \in A, f \in H^*$

$\delta =$ 

$: \quad R_H(A) \to H \otimes R_H(A)$

2. The quantum K-matrix is given by
$$K = K_{\text{ref}}(A) = \sum_i h_i \otimes 1_A \otimes f_i \in H \otimes A \rtimes (\widehat{H}^*)^{\text{op}},$$
where $\{h_i\}_i$ is a basis for H and $\{f_i\}$ the dual basis for H^*.

UNIVERSALITY

Definition

Let ${}^H\mathbf{QT}$ be the category of quasitriangular left H-comodule algebras.

(a) Objects are pairs, (Q, K), where Q is a left H-comodule algebra, and $K := K(Q) \in H \otimes Q$ is a quantum K-matrix for Q, and

(b) A morphism from (Q_1, K_1) to (Q_2, K_2) is a linear map $\phi : Q_1 \to Q_2$ that is both a left H-comodule morphism and an algebra morphism, such that $K_2 = (\mathrm{Id}_H \otimes \phi)(K_1)$.

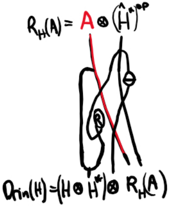
Theorem (L.–Walton–Yakimov)

When H is a finite-dimensional quasitriangular Hopf algebra over $\Bbbk$, we have that $(R_H(\Bbbk) = (\widehat{H}^*)^{\mathrm{op}}, K_{\mathrm{ref}}(\Bbbk))$ is an initial object of ${}^H\mathbf{QT}$.

CONCLUDING REMARKS

- A similar universal property for $\widehat{H}$, also called the the reflection equation algebra has been stated by Ben-Zvi–Brochier–Jordan [BZBJ18].
- The formalism here is analogue to a result by Radford for quasitriangular Hopf algebra [Rad94].
- The algebra $R_H(A)$ is a $\mathrm{Drin}(H)$-comodule algebra via coaction $\delta\colon R_H(A) \to \mathrm{Drin}(H) \otimes R_H(A)$

$$\delta(af) = \langle f_{(1)}, R^{(2)} \rangle a^{(-1)} R^{(1)} \otimes S^{-1}(f_{(3)}) \otimes a^{(0)} f^{(2)} =$$



- The H-comodule structure is recovered by restriction along the Hopf algebra map $\mathrm{Drin}(H) \to H$ obtained from R.

Thank you very much for your attention!

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