Categorification of cyclotomic rings

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AIM: p-DG Theory

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Summary

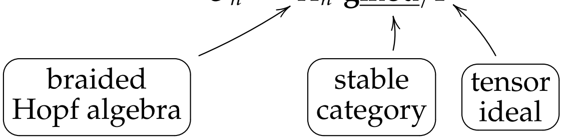
Summary:

- Hopfological algebra is a generalization of *homological algebra* using p-complexes, p a prime.
- $p = 2$ recovers classical homological algebra
- an approach to categorification of *small* quantum groups

Joint work with You Qi: ArXiv: `Math.QA/1804.01478`

We construct a triangulated tensor category

$$\mathcal{O}_n = H_n\text{-}\underline{\mathbf{gmod}}/\mathbf{I}$$



which categorifies the *cyclotomic integers* $\mathbb{O}_n = \mathbb{Z}[q]/\Phi_n(q)$.

Contents

BACKGROUND

Quantum groups:

- $\mathrm{U}_q(\mathfrak{g})$, q-deformation of $\mathrm{U}(\mathfrak{g})$ (Drinfeld, Jimbo)
- small quantum group: $\mathrm{u}_q(\mathfrak{g})$ finite-dimensional quotient, for q root of unity
- Categories $\dot{\mathrm{U}}_q(\mathfrak{g})$, $\dot{\mathrm{u}}_q(\mathfrak{g})$, objects are idempotents to ensure lattice grading (Lusztig)

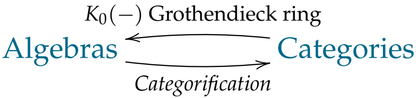
$\sim$**1990:** Reshetikhin–Turaev Invariants and 3D TQFT

- use representation theory of small quantum groups $u_q(\mathfrak{g})$
- $q^n = 1$ root of unity $\longleftrightarrow$ n is the level in Chern–Simons theory

BACKGROUND

∼**1994:** Categorification vision of Crane–Frenkel

- replace $\mathrm{u}_q(\mathfrak{g})$ by a *category* & study categorical representations
- the original algebra can be recovered via K_0 — the *Grothendieck ring*
- the *Categorification Program*



- *ultimate goal:* algebraic construction of invariants of 4D manifolds & 4D TQFT

CATEGORIFICATION

Progress: for q generic

- $\mathbb{Z}[q, q^{-1}]$-algebras, categorified using (graded) complexes
  $q \longleftrightarrow$ grading shift
- *Khovanov Homology* categorifies the *Jones Polynomial* [Khovanov ∼2000]
- categorification of $\dot{\mathrm{U}}_q(\mathfrak{g})$ [Khovanov–Lauda, Chuang–Rouquier 2005–2010]
- vast literature…

Categorification

Progress: for $q^n = 1$, a root of unity

- $\mathrm{u}_q(\mathfrak{g})$ is an $\mathbb{O}_n$-algebra — $\mathbb{O}_n$ cyclotomic integers
- Khovanov 2005: to categorify $\Phi_p(q) = 1 + q + \ldots + q^{p-1} = 0$ use the **stable category** of H-modules
- H a finite-dimensional Hopf algebra $\Rightarrow$ *Hopfological Algebra*
- categorifications of $\dot{\mathrm{u}}_q(\mathfrak{sl}_2^+)$ [Khovanov–Qi], $\dot{\mathrm{u}}_q(\mathfrak{sl}_2)$, $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$ [Elias–Qi] ∼2014–2016
- categorifications of Burau representation, tensor products of fundamental and Weyl representation [Qi–Sussan, Khovanov–Qi–Sussan]

Hopfological Algebra

H — finite-dimensional Hopf algebra over $\Bbbk$

- The regular H module has a 1-dimensional submodule $\iff$ an *integral element* $\Lambda \in H$ [Larson–Sweedler, 1960s]
- H is a *Frobenius algebra* (self-dual as an algebra)
- {f.d. injective modules} = {f.d. projective modules}

HOPFOLOGICAL ALGEBRA

A generalization of homological algebra:

- Happel defined a triangulated category: the *stable category* H-$\underline{\mathbf{mod}}$
- H-$\underline{\mathbf{mod}}$ and H-$\mathbf{mod}$ have the same objects
- In H-$\underline{\mathbf{mod}}$, projective modules P are annihilated ($\mathrm{id}_P = 0$)
- Short exact sequence:

$$0 \longrightarrow \Bbbk \xrightarrow{\Lambda} H \longrightarrow H/\Lambda H \longrightarrow 0$$

Tensoring with $H/\Lambda H \longleftrightarrow$ *Hopfological* shift

EXAMPLES

Example

$H = \Bbbk[\mathrm{d}]/(\mathrm{d}^2)$ is a Hopf algebra in super-vector spaces.
H-**gmod** gives (graded) chain complexes of $\Bbbk$-modules
H-$\underline{\textbf{gmod}}$ is the *derived category* of chain complexes

Example

Let $\operatorname{char} \Bbbk = p$. Then $H = \Bbbk[\mathrm{d}]/(\mathrm{d}^p)$ is a Hopf algebra over $\Bbbk$

$$\Delta(\mathrm{d}^n) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{d}^k \otimes \mathrm{d}^{n-k}, \qquad \Lambda = \mathrm{d}^{p-1}$$

- H-**mod** gives *p-complexes*
- Go back to Meyer's (1940s)

CATEGORIFIED CYCLOTOMIC INTEGERS

Challenge: Find $V \in H$-$\mathbf{g\underline{mod}}$ with

$$0 = [V] = 1 + q + \ldots + q^{p-1} = \Phi_p(q) \in K_0(H\text{-}\mathbf{g\underline{mod}}).$$

$1 \leftrightarrow$ trivial module

$q \leftrightarrow$ grading shift

Solution [Khovanov]: Use $H = \Bbbk[\mathrm{d}]/(\mathrm{d}^p)$ as a Hopf algebra if $\operatorname{char} \Bbbk = p$.
The regular module is projective:

$$0 = [H] = 1 + q + \ldots + q^{p-1} = \Phi_p(q) \in K_0(H\text{-}\mathbf{g\underline{mod}}).$$

$\Longrightarrow K_0(H\text{-}\mathbf{g\underline{mod}}) \cong \mathbb{O}_p$ — for a prime p.

p-DG ALGEBRAS

Hopfological algebra:

- study H-module algebras A — with a *p-differential* $\mathrm{d}\colon A \to A$, satisfying $\mathrm{d}^p = 0$, and $\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b)$
- triangulated category A-$\underline{\mathbf{gmod}}_H$ — quotient of A-$\mathbf{mod}$
- $K_0(A\text{-}\underline{\mathbf{gmod}}_H)$ categorifies $\mathbb{O}_p$-modules (or $\mathbb{O}_p$-algebras)

Theorem (Elias–Qi, 2013)

The diagrammatic category $U = \bigoplus_{\lambda,\mu\in\mathbb{Z}} U^{\lambda}$ *(Lauda) equipped with a p-differential categorifies the idempotent version of the small quantum group* $\dot{\mathrm{u}}_q(\mathfrak{sl}_2)$*, i.e.*

$$\bigoplus_{\lambda\in\mathbb{Z}} K_0(\mathrm{D}_H(U^{\lambda})\text{-}\mathbf{mod}) \cong \dot{\mathrm{u}}_{\mathbf{q}}(\mathfrak{sl}_{\mathbf{2}})$$

- Elias–Qi (2015) also categorified $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$

A BRAIDED HOPF ALGEBRA

Question: What if $n \neq p$?
Consider polynomial ring $\Bbbk[\mathrm{d}]$, $\Delta(\mathrm{d}) = \mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}$

$$\Delta(\mathrm{d}^n) = \sum_i \binom{n}{i} \mathrm{d}^{n-i} \otimes \mathrm{d}^i$$

(d^n) is *no* Hopf ideal.

Alternative: View $\Bbbk[\mathrm{d}]$ as a *braided* Hopf algebra in $\mathbf{gVec}_q$: $\mathbb{Z}$-graded vector spaces, braiding $\Psi(v \otimes w) = q^{|v||w|} w \otimes v$ — for fixed $q^n = 1$ primitive

$$\Delta(\mathrm{d}^n) = \sum_i \binom{n}{i}_q \mathrm{d}^{n-i} \otimes \mathrm{d}^i = \mathrm{d}^n \otimes 1 + 1 \otimes \mathrm{d}^n$$

$\Longrightarrow \Bbbk[\mathrm{d}]/(\mathrm{d}^n)$ is a Hopf algebra in $\mathbf{gVec}_q$

MULTIPLE p-DIFFERENTIALS

For general n, Grothendieck ring is too large:

$$K_0(\Bbbk[\mathrm{d}]/(\mathrm{d}^n)\text{-}\underline{\mathbf{gmod}}) \cong \mathbb{Z}[q, q^{-1}]/(1 + q + \ldots + q^{n-1}) \not\cong \mathbb{O}_n.$$

Modules are n-complexes of [Meyer 1940s], [Kapranov ~1996]

Mirmohades 2015: Categorification of $\mathbb{O}_{pq}$ ($p \neq q$ odd primes) using quotient by an ideal in stable category of tensor product of two *Taft algebras* $(\Bbbk[\mathrm{d}]/(\mathrm{d}^p) \rtimes \Bbbk C_p) \otimes (\Bbbk[\mathrm{d}]/(\mathrm{d}^q) \rtimes \Bbbk C_q)$.

More general example: $n = 2^2 \cdot 3$, need modules V_1, V_2 with

$$[V_1] = 1 + q^6, \qquad [V_2] = 1 + q^4 + q^8$$

$$\Bbbk \xrightarrow{\mathrm{d}_1} \Bbbk\{-6\} \qquad \Bbbk \xrightarrow{\mathrm{d}_2} \Bbbk\{-4\} \xrightarrow{\mathrm{d}_2} \Bbbk\{-8\}$$

$$\gcd(1 + q^6, 1 + q^4 + q^8) = \Phi_6(q^2) = \Phi_{12}(q) = 1 - q^2 + q^4$$

THE BRAIDED HOPF ALGEBRA H_n

Idea: Use *multiple differentials* of different degrees and order, which commute.

Definition (L.–Qi)

Decompose $n = p_1^{a_1} \dots p_t^{a_t}$, for p_k pairwise distinct primes

$$H_n := \Bbbk[\mathrm{d}_1, \dots, \mathrm{d}_t]/(\mathrm{d}_1^{p_1}, \dots \mathrm{d}_t^{p_t}), \qquad \deg(\mathrm{d}_k) = n_k = n/p_k$$

$\Longrightarrow H_n$ is a Hopf algebra in $\mathbf{gVec}_q$ with $\Delta(\mathrm{d}_k) = \mathrm{d}_k \otimes 1 + 1 \otimes \mathrm{d}_k$

HOPF SUBALGEBRAS $\widehat{H}_n^k$

Definition

Hopf subalgebras $\widehat{H}_n^k$: Consider the subalgebra generated by all differentials besides d_k:

$$\widehat{H}_n^k := \frac{\Bbbk[\mathrm{d}_1, \ldots, \widehat{\mathrm{d}_k}, \ldots \mathrm{d}_t]}{(\mathrm{d}_1^{p_1}, \ldots, \widehat{\mathrm{d}_k^{p_k}}, \ldots \mathrm{d}_t^{p_t})}$$

$\implies$ $\widehat{H}_n^k$ is Hopf subalgebra and quotient algebra of H_n.

Modules W_k: W_k is the free $\widehat{H}_n^k$-module regarded as an H_n-module, where d_k acts by zero, d_l act freely for $l \neq k$.

The ideal I

Definition (The ideal I)

Define $\mathbf{I}_k$ as the full subcategory on objects $V \in H_n$-$\underline{\mathbf{gmod}}$ which are images of H_n-modules with a filtration by objects $W_k\{b\}$.
Let $\mathbf{I}$ be the full subcategory of H_n-$\underline{\mathbf{gmod}}$ which consists of objects $U = \bigoplus_{k=1}^{t} U_k$, where U_k is an object in $\mathbf{I}_k$.

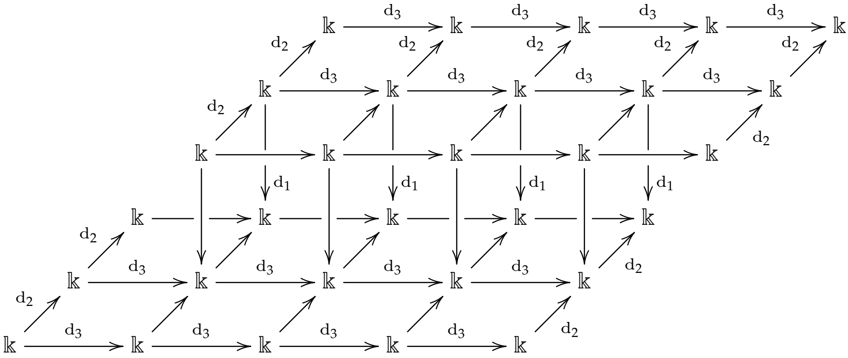
Notes:

- There are no extensions between objects of $\mathbf{I}_k$ and $\mathbf{I}_l$ for $k \neq l$ (besides free modules, which are zero in H_n-$\underline{\mathbf{gmod}}$).
- In fact, all morphisms from objects in $\mathbf{I}_k$ to objects in $\mathbf{I}_l$ are null-homotopic.
- If $n = p^a$ then $\mathbf{I}$ is the zero ideal in H_n-$\underline{\mathbf{gmod}}$.

THE IDEAL I

Example

For $n = 2^a \cdot 3^b \cdot 5^c$, there exist various non-split extensions in $\mathbf{I}_1$, e.g.:



Main Result

Theorem (L.–Qi)

- $\mathbf{I}$ *is a thick (or saturated) tensor ideal in* H_n-$\underline{\mathbf{gmod}}$
- *There is an isomorphism of algebras*

$$K_0(H_n\text{-}\underline{\mathbf{gmod}}/\mathbf{I}) \cong \mathbb{O}_n$$

Verdier quotient

Elements of the proof

Decompose $n = p_1^{a_1} \dots p_t^{a_t}$, pairwise distinct primes $p_1, \dots, p_t$. Denote $m = p_1 \dots p_t$, the **radical** of n.

Lemma

The Grothendieck group of H_n-$\underline{\mathbf{gmod}}$ is isomorphic, as a $\mathbb{Z}[q, q^{-1}]$-algebra, to the quotient ring

$$K_0(H_n\text{-}\underline{\mathbf{gmod}}) \cong \frac{\mathbb{Z}[q, q^{-1}]}{\langle \prod_{k=1}^{t} \frac{[m]_\nu}{[m/p_t]_\nu} \rangle}, \qquad \textit{where } \nu = q^{n/m}.$$

The tensor product on H_n-$\mathbf{gmod}$ descends to the multiplication on the Grothendieck group level, while the grading shift functor $\{1\}$ descends to multiplication by q.

Note:

$$\frac{[m]_\nu}{[m/p_k]_\nu} = 1 + q^{n_k} + \dots + q^{(p_k - 1)n_k}$$

ELEMENTS OF THE PROOF

Again, for $n = p_1^{a_1} \dots p_t^{a_t}$ and $m = p_1 \dots p_t$, observe that

$$\Phi_n(q) = \Phi_m(q^{n/m}) = \gcd\left(\frac{[m]_\nu}{[m/p_1]_\nu}, \dots, \frac{[m]_\nu}{[m/p_t]_\nu}\right), \quad \text{for } \nu = q^{n/m}$$

And for the modules W_k generating $\mathbf{I}$:

$$[W_k] = \prod_{l \neq k} \frac{[m]_\nu}{[m/p_k]_\nu} = \prod_{l \neq k} (1 + q^{n_k} + \dots + q^{(p_k - 1)n_k})$$

$$\implies \qquad \Phi_n(q) = \gcd\left([W_1], \dots, [W_t]\right)$$

$$\implies K_0(\mathbf{I}) = \langle \Phi_n(q) \rangle.$$

CONCLUDING REMARKS

- as an algebra, H_n is independent of q, but the coproduct depends on q
- natural isomorphisms $V \otimes_q W \xrightarrow{\sim} W \otimes_{q^{-1}} V$ — no braiding
- Galois action $q \mapsto q^a$, there are Hopf algebra isomorphisms of bosonizations $H_n \rtimes_q \Bbbk C_n \cong H_n \rtimes_{q^a} \Bbbk C_n$, for $a \in (\mathbb{Z}/n\mathbb{Z})^\times$
- H_n-**gmod** is a *spherical category*

Thank you for your attention!

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