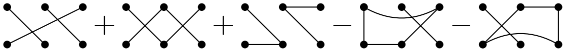
Interpolation Categories, Centers and Link Invariants

$c_1 =$ 

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SUMMARY

Reference: Arxiv:1901.08657

Summary: We construct braided monoidal subcategories

$$\mathcal{D}_t \hookrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}(S_t))$$

Monoidal center

Deligne's interpolation category of $\mathrm{Rep}(S_n)$, where $t \in \mathbb{C}$

- $\mathcal{D}_t$ is a ribbon category
- For $n \in \mathbb{N}$, $\mathcal{D}_n \longrightarrow\!\!\!\!\!\rightarrow \mathcal{Z}(\mathrm{Rep}(S_n))$ is essentially surjective & full
- Application: Invariants of framed links, polynomial in t

Contents

YETTER–DRINFELD MODULES & DIJKGRAAF–WITTEN THEORY

G a finite group, $\operatorname{char}\Bbbk = 0$
A *Yetter–Drinfeld module* over G is a G-graded G-module

$$V = \bigoplus_{g\in G} V_g, \qquad \text{such that } h\cdot V_g = V_{hgh^{-1}}.$$

- YD modules over G form a modular tensor category
- invariants of links $\mathcal{L}$ and 3-manifolds $C = \mathbb{R}^3 \setminus \overline{\mathcal{L}}$

$$Z_G^{\mathrm{DW}}(C) = \frac{1}{|G|} \underbrace{\left|\mathrm{Hom}_{\mathrm{group}}(\pi_1(C), G)\right|}_{\mathrm{Inv}_G^{\mathrm{DW}}(\mathcal{L})}$$

Dijkgraaf–Witten theory: A fully extended 3D TQFT $Z_{G,\omega}^{\mathrm{DW}}$
Here: $1 = \omega \in H^3(G, \Bbbk^\times)$ — the *untwisted* case

DELIGNE'S INTERPOLATION CATEGORY

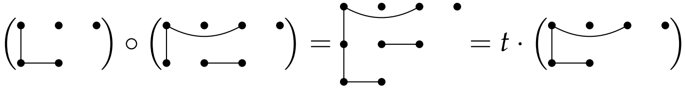
Motivation: Let $\mathfrak{h} := \mathbb{C}^n$ *standard S_n-representation*.

- Every simple S_n-representation is a direct summand of $\mathfrak{h}^{\otimes k}$ for some $k \geq 0$.
- Partitions of $\{1, \ldots, k, 1', \ldots, l'\}$ give morphisms of S_n-representations
$$\mathfrak{h}^{\otimes k} \to \mathfrak{h}^{\otimes l}$$
- These morphisms span $\mathrm{Hom}_{S_n}(\mathfrak{h}^{\otimes k}, \mathfrak{h}^{\otimes l})$ as a $\mathbb{k}$-vector space.
- $\mathrm{Rep}(S_n)$ is the *idempotent completion* (the *Karoubian envelope*) of the full tensor subcategory generated by $\mathfrak{h}$.
- Deligne: Composition rule is combinatorial, the number n appears "polynomially".
- replacing n by$t \in \mathbb{C}$ gives new tensor categories $\mathrm{Rep}(S_t)$

Deligne's Interpolation Category

$\underline{\mathrm{Rep}}(S_t)$ is the *idempotent completion* of $\underline{\mathrm{Rep}}^0(S_t)$ which has:

- Objects: $[m]$ for $m \in \mathbb{Z}_{\geq 0}$
- Morphisms $[m] \to [k]$: Partitions of $\{1, \ldots, m, 1', \ldots, k'\}$
- Composition: Concatenation — for example,



Deligne '07: Symmetric monoidal category $\underline{\mathrm{Rep}}(S_t)$ for $t \in \Bbbk$

- For *generic* $t \notin \mathbb{Z}_{\geq 0}$: $\underline{\mathrm{Rep}}(S_t)$ is semisimple
- For $n \in \mathbb{N}$:

$$\mathcal{F}_n : \underbrace{\underline{\mathrm{Rep}}(S_n)}_{\text{not semisimple}} \longrightarrow\!\!\!\!\!\rightarrow \underbrace{\mathrm{Rep}(S_n)}_{\text{semisimplification}}$$

is full & essentially surjective

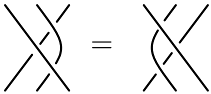
THE MONOIDAL CENTER

Drinfeld, Majid, Joyal–Street:

$\mathcal{C}$ monoidal category $\Longrightarrow \mathcal{Z}(\mathcal{C})$ a *braided* monoidal category

- Objects: (V, c), $V \in \mathcal{C}$, *half-braiding* $c_W \colon V \otimes W \to W \otimes V$, natural in W, such that

$$c_{W \otimes U} = (\mathrm{Id}_W \otimes c_U)(c_W \otimes \mathrm{Id}_U) \Longrightarrow$$



  - (V, c_V) is a solution to the Quantum Yang–Baxter Equation
- Morphisms: required to commute with the half-braidings

Goals:

- Obtain an interpolation category for Yetter–Drinfeld modules over S_n ✓
- Classify all objects in $\mathcal{Z}(\underline{\mathrm{Rep}}(S_t))$ **(work in progress)**

INTERPOLATION OBJECTS

All *simple* Yetter–Drinfeld modules over S_n are:

$$\{W_{\mu,V} \mid \mu \vdash n,\ V \text{ simple } Z(\mu)\text{-module}\}$$

- $Z(\mu)$ is the centralizer of $\sigma \in S_n$ of cycle type μ
- $W_{\mu,V} \cong \mathrm{Ind}_{Z(\mu)}^{S_n}(V)$ as an S_n-module

Proposition (Flake–L.)

Given μ, V as above, construct in $\underline{\mathrm{Rep}}(S_t)$:

- *an idempotent $e_V : [n] \to [n]$*
- *a morphism $c_1^V : ([n], e_V) \otimes [1] \to [1] \otimes ([n], e_V)$*

$\Rightarrow$ These determine an interpolation object $\underline{W}_{\mu,V}$ in $\mathcal{Z}(\underline{\mathrm{Rep}}(S_t))$.

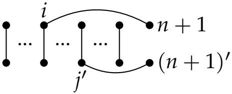
Interpolation Objects

Data: $n \geq 1$, $\mu \vdash n$, $\sigma \in S_n$ of cycle type μ,

$\rho\colon Z(\mu) \to \mathrm{Mat}_{k\times k}(\mathbb{k})$ simple representation V

Interpolation Object: Define $\underline{W}_{\mu,V} = \big(([n]^{\oplus k}, e_\rho),\, c^\mu\big)$ in $\mathcal{Z}(\underline{\mathrm{Rep}}(S_t))$:

$$e_\rho = \frac{1}{|Z(\mu)|}\sum_{z\in Z(\mu)} x_z \otimes \rho(z)$$

$E_j^i =$ 

$$c^\mu_{[1]} = \Psi^{\oplus k}_{[n],[1]}\left(\mathrm{Id}_{[n+1]} + \sum_{i=1}^{n}\left(E^i_{\sigma(i)} - E^i_i\right)\right)^{\otimes k}(e_\rho \otimes \mathrm{Id}_{[1]})$$

INTERPOLATION OBJECTS

Proposition (Flake–L.)

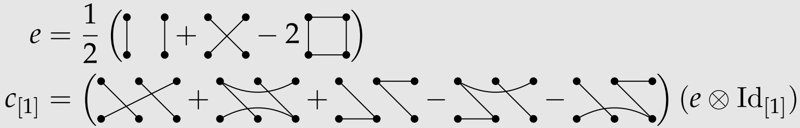
Let $n \in \mathbb{N}$. For the induced functor

$$\mathcal{F}_n \colon \mathcal{Z}(\underline{\mathrm{Rep}}(S_n)) \longrightarrow \mathcal{Z}(\mathrm{Rep}(S_n))$$

we have $\mathcal{F}_n(\underline{W}_{\mu,V}) \cong W_{\mu,V}$ as a Yetter–Drinfeld modules over S_n.

Example

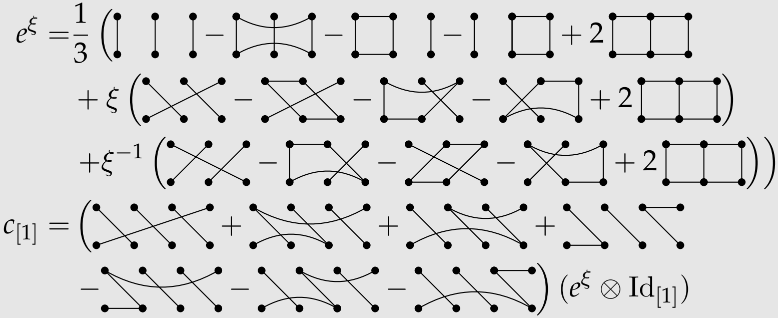
For $\mu = (2) \vdash 2$, $Z(\mu) = \mathbb{Z}_2$, $V = \Bbbk^{\mathrm{triv}}$, the object $\underline{W}_{(2),\Bbbk^{\mathrm{triv}}}$ has



Interpolation Objects

Example

For $\mu = (3) \vdash 3$, $Z(\mu) = \mathbb{Z}_3$, irreducible modules V^ξ, ξ third root of unity, the object $\underline{W}_{(3),V^\xi}$ has



The Category $\mathcal{D}_t$

Definition

Let $\mathcal{D}_t$ denote the idempotent completion of the full subcategory of $\mathcal{Z}(\underline{\mathrm{Rep}}(S_t))$ generated by *all* interpolation objects $\underline{W}_{\mu,V}$.

Theorem (Flake–L.)

For $n \in \mathbb{Z}_{\geq 0}$, the functor

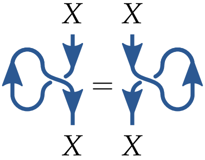
$$\mathcal{F}_n\colon \mathcal{D}_n \twoheadrightarrow \mathcal{Z}(\mathrm{Rep}(S_n))$$

of braided monoidal categories is essentially surjective and full on morphism spaces.

$\mathcal{D}_t$ is a Ribbon Category

A *ribbon* category is a braided monoidal category with two-sided duals (i.e. a *pivotal* category) in which

$\theta^l_X =$



$= \theta^r_X,$

for any object X, i.e. left and right *twists* are equal.

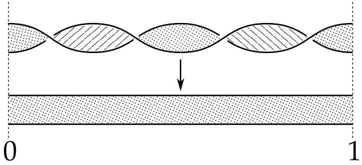
Theorem (Flake–L.)

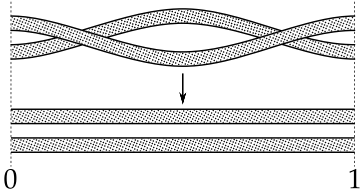
$\mathcal{D}_t$ is a ribbon category.
For $W = \underline{W}_{\mu,V}$, $k = \dim V$, the left and right twists are given by

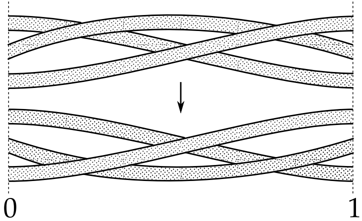
$$\theta^l_W = \theta^r_W = (\sigma^{-1})^{\oplus k} e_V, \qquad \textit{where } \sigma \textit{ has cycle type } \mu.$$

Framed Ribbon Links

- Let $\mathcal{L}$ be a *framed ribbon link*, i.e. an oriented link with ribbons instead of strings.
- Two framed ribbon links are *equivalent* if related through three *Reidemeister moves*:







Framed Ribbon Link Invariants

- The category of *framed ribbon tangles* is a *free* ribbon category
- Every object X in a ribbon category provides an invariant $\mathrm{Inv}_X(\mathcal{L})$ of framed ribbon links [Reshetikhin–Turaev]
- The category $\mathcal{Z}(\mathrm{Rep}(G))$ gives the untwisted Dijkgraaf–Witten invariants

Corollary

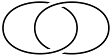
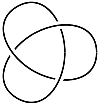
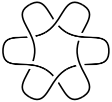
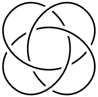
Let $\mu \vdash n$. Given an interpolation object $\underline{W}_{\mu,V}$ in $\mathcal{D}_t$, the polynomial

$$\mathrm{P}_{\mu,V}(\mathcal{L},t) := \mathrm{Inv}_{\underline{W}_{\mu,V}}(\mathcal{L}) \in \Bbbk[t]$$

is an invariant of framed ribbon links.

The evaluation $\mathrm{P}_{\mu,V}(\mathcal{L},n)$ recovers the corresponding untwisted Dijkgraaf–Witten invariant.

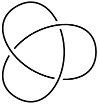
Examples of Ribbon Link Polynomials

Ribbon torus link $\mathcal{T}$	$\frac{P_{(2),\Bbbk^{\mathrm{triv}}}(\mathcal{T},t)}{\dim \underline{W}_{(2),\Bbbk^{\mathrm{triv}}}}$	$\frac{P_{(3),\Bbbk^{\mathrm{triv}}}(\mathcal{T},t)}{\dim \underline{W}_{(3),\Bbbk^{\mathrm{triv}}}}$
$\mathcal{T}_{2,-2} =$ 	$\frac{t^2}{2} - \frac{5t}{2} + 4$	$\frac{t^3}{3} - 4t^2 + \frac{47t}{3} - 18$
$\mathcal{T}_{2,-3} =$ 	$2t - 3$	$3t - 8$
$\mathcal{T}_{2,-6} =$ 	$\frac{t^2}{2} - \frac{t}{2}$	$\frac{t^3}{3} - 4t^2 + \frac{56t}{3} - 27$
$\mathcal{T}_{3,-4} =$ 	$2t^2 - 8t + 9$	$3t^3 - 36t^2 + 144t - 188$

$$\dim \underline{W}_{(2),\Bbbk^{\mathrm{triv}}} = \tfrac{1}{2}t(t-1), \ \dim \underline{W}_{(3),\Bbbk^{\mathrm{triv}}} = \tfrac{1}{3}t(t-1)(t-2)$$

Some More Trefoil Invariants

The *left-handed trefoil link*

$\mathcal{T}_{2,-3}=$ 

Cycle type μ	$\dfrac{\mathrm{P}_{\mu,\Bbbk\mathrm{triv}}(\mathcal{T}_{2,-3},t)}{\dim \underline{W}_{\mu,\Bbbk\mathrm{triv}}}$
(1)	1
(2)	$2t-3$
(3)	$3t-8$
(4)	$2t^2-16t+37$
$(2,2)$	$4t^2-28t+49$

Further Questions

- Effective computation of the ribbon link polynomials, currently computed using *Wolfram Mathematica*®
- $\mathcal{D}_t$ is non-semisimple for $t \in \mathbb{Z}_{\geq 0}$
  Is $\mathcal{D}_t$ semisimple (like $\underline{\text{Rep}}(S_t)$) if t is generic? ✓ Yes
- Is $\mathcal{D}_t \simeq \mathcal{Z}(\underline{\text{Rep}}(S_t))$? (work in progress)
- Applications to invariants of 3-manifolds and TQFT?
- Can anything be done in the *twisted* case?

Thank you for your attention!