Frobenius monoidal functors on Drinfeld centers

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Joint work with Johannes Flake (Bonn) and Sebastian Posur (Münster)
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Outline

Motivation — Morphisms of centers

Classical problem:

- $f\colon R \to S$ is a morphism of rings
- There is no restriction to a map $Z(R) \to Z(S)$ in general

Categorical analogues:

- Ring $(A, A \times A \xrightarrow{m} A, 1_A) \rightsquigarrow$ monoidal category $(\mathcal{C}, \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}, \mathbb{1}_{\mathcal{C}})$
- Center $Z(A) \rightsquigarrow$ *Drinfeld center* $\mathcal{Z}(\mathcal{C})$
- Morphism of rings $\rightsquigarrow$ (strong) monoidal functor $G\colon \mathcal{C} \to \mathcal{D}$

$$\mathrm{lax}^G_{A,B}\colon G(A) \otimes G(B) \underset{\sim}{\overset{\sim}{\rightleftarrows}} G(A \otimes B)\colon \mathrm{oplax}^G_{A,B} \quad + \quad \text{coherences...}$$

Theorem (Flake–L.–Posur)

Under certain conditions, an ambiadjoint F of G induces a braided Frobenius monoidal functor $\mathcal{Z}(F)\colon \mathcal{Z}(\mathcal{D}) \to \mathcal{Z}(\mathcal{C})$.

Some motivating examples

- $\phi\colon \mathsf{H} \hookrightarrow \mathsf{G}$ finite groups, $\omega \in H^3(\mathsf{G}, \Bbbk^\times)$ 3-cocycle,

$$\mathcal{Z}(\mathrm{Rep}\,\mathsf{H}) \to \mathcal{Z}(\mathrm{Rep}\,\mathsf{G}) \qquad \text{[Flake–Harman–L.]}$$

$$\mathcal{Z}(\mathbf{Vect}_{\mathsf{H}}^{\phi^*\omega}) \to \mathcal{Z}(\mathbf{Vect}_{\mathsf{G}}^{\omega}) \qquad \text{[Hannah–L.–Ros Camacho]}$$

  braided Frobenius monoidal functors

- Application: classifying connected étale algebras in $\mathcal{Z}(\mathbf{Vect}_{\mathsf{G}}^{\omega})$ [Davydov, Davydov–Simmons, L.–Walton, H.–L.–R.C.]
- For all $n \in \mathbb{Z}_{\geq 0}$, $t \in \mathbb{C}$,

$$\underline{\mathrm{Ind}}\colon \mathcal{Z}(\mathrm{Rep}\,S_n) \longrightarrow \mathcal{Z}(\underline{\mathrm{Rep}}\,S_t)$$

  braided Frobenius monoidal functor [Flake–Harman–L.]

- Application: classify indecomposable objects in $\mathcal{Z}(\underline{\mathrm{Rep}}\,S_t)$ [F.–H.–L.]

This talk: General results on Frobenius monoidal functors on Drinfeld centers

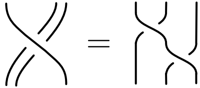
Background — The Drinfeld Center

$\mathcal{C}$ monoidal category $\rightsquigarrow$ Drinfeld center $\mathcal{Z}(\mathcal{C})$:

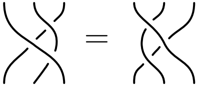
- Objects of $\mathcal{Z}(\mathcal{C})$: Pairs (V, c^V), $V \in \mathcal{C}$, *half-braiding*

$$c^V_W = \times : V \otimes W \to W \otimes V,$$

$$c^V_{W\otimes U} = (\mathrm{id}_W \otimes c^V_U)(c^V_W \otimes \mathrm{id}_U) \quad \Leftrightarrow$$



$\Rightarrow (V, c^V_V)$ solution of Quantum Yang–Baxter Equation



- Morphisms of $\mathcal{Z}(\mathcal{C})$: morphisms in $\mathcal{C}$ commuting with half-braidings

Theorem (Drinfeld, Majid, Joyal–Street ∼1990)

For $\mathcal{C}$ a tensor category, $\mathcal{Z}(\mathcal{C})$ is a braided tensor category.

The braiding Ψ is obtained from the half-braidings: $\Psi_{V,W} = c^V_W$.

The Drinfeld Center — Examples

Modules over a finite-dimensional Hopf algebra H, $\mathcal{C} = H\text{-}\mathbf{Mod}$
$\implies \mathcal{C}$ is a tensor category, with $\otimes$ via *coproduct* $\Delta\colon H \to H \otimes_{\Bbbk} H$

Question: What is the center $\mathcal{Z}(\mathcal{C})$ in this case?

Answer 1: Modules over the Drinfeld double $\mathrm{Drin}(H)$, a Hopf algebra $\mathrm{Drin}(H)$ on $H \otimes_{\Bbbk} H^*$ with H, H^* Hopf subalgebras.

Example

$H = \Bbbk\mathsf{G}$ *a group algebra,* $|\mathsf{G}| < \infty$. *Then* $\mathrm{Drin}(\mathsf{G})$ *is defined on* $\Bbbk\mathsf{G} \otimes \Bbbk[\mathsf{G}]$,

$$g\delta_h = \delta_{ghg^{-1}} g, \qquad \forall g, h \in \mathsf{G}.$$

More generally, twist by a 3-cocycle $\omega \rightsquigarrow \mathrm{Drin}^{\omega}(\mathsf{G})$ *[Dijkgraaf–Witten theory]*

- **Applications:** Construction of modular tensor categories, 3D TQFTs
- For G algebraic group, $\mathcal{Z}(\mathrm{Rep}\,\mathsf{G}) \simeq \mathcal{O}_{\mathsf{G}}\text{-}\mathbf{Mod}_{\mathrm{Rep}\,\mathsf{G}} =: \mathbf{QCoh}(\mathsf{G}/^{\mathrm{ad}}\mathsf{G})$

Yetter–Drinfeld Modules

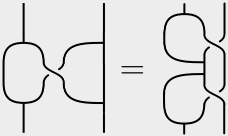
Question: What is the center $\mathcal{Z}(\mathcal{C})$ for $\mathcal{C} = H\text{-}\mathbf{Mod}$?

Answer 2: The category of Yetter–Drinfeld modules ${}^{H}_{H}\mathbf{YD}$.

Definition

Yetter–Drinfeld modules (V, a, δ) over H.

- $a =$ : $H \otimes V \to V$ makes V an H-module
- $\delta =$ : $V \to H \otimes V$ makes V an H-comodule
- Compatibility: *Yetter–Drinfeld condition*



Proposition

For a Hopf algebra H and $\mathcal{C} = H\text{-}\mathbf{Mod}$, $\mathcal{Z}(\mathcal{C}) \simeq {}^{H}_{H}\mathbf{YD}$.

Drinfeld center of bimodules

More generally: $\mathcal{C}$ monoidal category, $\mathcal{M}$ a $\mathcal{C}$-bimodule,

$$\triangleright \colon \mathcal{C} \times \mathcal{M} \to \mathcal{M}, \qquad \triangleleft \colon \mathcal{M} \times \mathcal{C} \to \mathcal{M}$$

Definition ($\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$, Gelaki–Naidu–Nikshych, Greenough, . . .)

- **Objects:** (M, c), where $M \in \mathcal{M}$ and c *half-braiding*, a natural isomorphism $c_A^M \colon M \triangleleft A \xrightarrow{\sim} A \triangleright M$ satisfying:
$$c^M_{A\otimes B} = (A \triangleright c^M_B)(c^M_A \triangleleft B)$$
- **Morphisms:** $f \colon (M, c^M) \to (N, c^N) \overset{\text{corresponds to}}{\longleftrightarrow} f \in \mathrm{Hom}_{\mathcal{M}}(M, N)$ s.t.:

$$\begin{array}{ccc} M \triangleleft A & \xrightarrow{\;c^M_A\;} & A \triangleright M \\ \downarrow{\scriptstyle f \triangleleft A} & & \downarrow{\scriptstyle A \triangleright f} \\ N \triangleleft A & \xrightarrow{\;c^N_A\;} & A \triangleright N. \end{array}$$

Centers of bimodules are 2-functorial

Special cases:

- $\mathcal{C}^{\mathrm{reg}}$ — the *regular* $\mathcal{C}$-bimodule, action via $\otimes$
  Then $\mathcal{Z}(\mathcal{C}) = \mathcal{Z}_{\mathcal{C}}(\mathcal{C}^{\mathrm{reg}})$ — the usual Drinfeld center of $\mathcal{C}$
- A *strong monoidal functor* $G\colon \mathcal{C} \to \mathcal{D}$ makes $\mathcal{D}$ a $\mathcal{C}$-bimodule, $\mathcal{D}^G$ — restricting $\mathcal{D}^{\mathrm{reg}}$ along G
- $\mathcal{Z}_{\mathcal{C}}(\mathcal{D}^G)$ is a *monoidal category* [Majid]

Proposition (2-Functoriality [Shimizu])

A $\mathcal{C}$-bimodule functor $F\colon \mathcal{M} \to \mathcal{N}$ induces a functor of categories

$$\mathcal{Z}_{\mathcal{C}}(F)\colon \mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \to \mathcal{Z}_{\mathcal{C}}(\mathcal{N}).$$

Bimodule transformation $\eta\colon F \to G$ gives a natural transformation $\mathcal{Z}_{\mathcal{C}}(\eta)\colon \mathcal{Z}_{\mathcal{C}}(F) \to \mathcal{Z}_{\mathcal{C}}(G)$ $\Longrightarrow$ *2-functor $\mathcal{Z}_{\mathcal{C}}\colon \mathcal{C}\text{-}\mathbf{BiMod} \to \mathbf{Cat}$*

Monoidal adjunctions

Define a **2-category** $\mathbf{Cat}^{\otimes}_{\text{lax}}$:

- Objects: monoidal categories
- 1-Morphisms: *lax* monoidal functors
- 2-Morphisms: monoidal natural transformations $\eta\colon F \to G$:

$$\begin{array}{ccc} F(X)\otimes F(Y) & \xrightarrow{\text{lax}^F_{X,Y}} & F(X\otimes Y) \\ \downarrow{\scriptstyle \eta_X\otimes\eta_Y} & & \downarrow{\scriptstyle \eta_{X\otimes Y}} \\ G(X)\otimes G(Y) & \xrightarrow{\text{lax}^G_{X,Y}} & G(X\otimes Y) \end{array} \qquad \begin{array}{ccc} & \mathbb{1} & \\ {\scriptstyle \text{lax}^G_0}\swarrow & & \searrow{\scriptstyle \text{lax}^F_0} \\ F(\mathbb{1}) & \xrightarrow{\eta_{\mathbb{1}}} & G(\mathbb{1}) \end{array}$$

Definition (Monoidal adjunction)

A monoidal adjunction $G \dashv R$ is an adjunction *internal* to $\mathbf{Cat}^{\otimes}_{\text{lax}}$.

- $G \dashv R$ monoidal adjunction $\implies$ G is *strong* monoidal
- G strong monoidal $\Rightarrow \exists!$ *lax* structure on R s.t. $G \dashv R$ is a monoidal adjunction [Kelly '74, doctrinal adjunction]

The projection formula morphisms

Definition (Projection formula morphisms)

$$\begin{array}{ccccc} A \otimes RX & & \xrightarrow{\mathrm{lproj}^R_{A,X}} & & R(GA \otimes X) \\ & \searrow^{\mathrm{unit}_A \otimes \mathrm{id}} & & \nearrow_{\mathrm{lax}_{GA,X}} & \\ & & RG(A) \otimes RX & & \end{array} \qquad \begin{array}{ccccc} RX \otimes A & & \xrightarrow{\mathrm{rproj}^R_{X,A}} & & R(X \otimes GA) \\ & \searrow^{\mathrm{id} \otimes \mathrm{unit}_A} & & \nearrow_{\mathrm{lax}_{X,GA}} & \\ & & RX \otimes RG(A) & & \end{array}$$

If lproj^R and rproj^R are invertible, say: the *projection formula holds* for R.

- In representation theory (*Frobenius reciprocity*): H ⊂ G finite groups, $\mathrm{Ind} \dashv \mathrm{Res}$ (op)monoidal adjunction,
$$\mathrm{lproj}_{V,W} \colon \mathrm{Ind}(\mathrm{Res}(V) \otimes W) \xrightarrow{\sim} V \otimes \mathrm{Ind}(W)$$
- In algebraic geometry: $f \colon X \to Y$ morphism of schemes, $f^* \dashv f_*$, $\mathcal{E} \in \mathbf{QCoh}(Y)$, $\mathcal{F} \in \mathbf{QCoh}(X)$ locally free,
$$\mathrm{lproj}_{\mathcal{E},\mathcal{F}} \colon \mathcal{E} \otimes_{\mathcal{O}_X} f_*(\mathcal{F}) \xrightarrow{\sim} f_*(f^*(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{F})$$

The projection formula morphisms

A sufficient criterion:

Proposition (Fausk–Hu–May, Flake–L.–Posur)

$\mathcal{C}$ rigid (left and right duals exist) $\Longrightarrow$ the projection formula holds for R

- More generally, if $\mathcal{C}$ has internal hom objects and G preserves them, then the projection formulas hold for R.
- For an opmonoidal adjunction $G \dashv L$, the projection formula morphisms

$$L(GA \otimes X) \xrightarrow{\mathrm{lproj}_{A,X}} A \otimes LX, \qquad L(A \otimes GA) \xrightarrow{\mathrm{rproj}_{X,A}} LX \otimes A$$

  are also called Hopf operators
- The monad $G \circ L$ is a *Hopf monad* if and only if the projection formulas hold for $L \dashv G$ [Bruguieres–Lack–Virelizier '11].

Categorical bimodule functors

Proposition (F.–L.–P.)

Let $G \dashv R$ be a monoidal adjunction.
projection formula $\implies$ morphism of $\mathcal{C}$-bimodules $R \colon \mathcal{D}^G \to \mathcal{C}$ with:

$$
\begin{array}{ccc}
R(A \triangleright X) & \xrightarrow{\mathrm{lin}^l_{A,X}} & A \triangleright RX \\
\| & & \| \\
R(GA \otimes X) & \xrightarrow{(\mathrm{lproj}_{A,X})^{-1}} & A \otimes RX
\end{array}
\qquad
\begin{array}{ccc}
R(X \triangleleft A) & \xrightarrow{\mathrm{lin}^r_{X,A}} & RX \triangleleft A \\
\| & & \| \\
R(X \otimes GA) & \xrightarrow{(\mathrm{rproj}_{X,A})^{-1}} & RX \otimes A
\end{array}
$$

Monoidal adjunction of categories $/\mathcal{C}$-bimodules:

$$
\mathcal{C} \underset{R}{\overset{G}{\rightleftarrows}} \mathcal{D}^G \ (G \dashv R) \quad \implies \quad \mathcal{Z}_{\mathcal{C}}(\mathcal{C}) \underset{\mathcal{Z}_{\mathcal{C}}(R)}{\overset{\mathcal{Z}_{\mathcal{C}}(G)}{\rightleftarrows}} \mathcal{Z}_{\mathcal{C}}(\mathcal{D}^G) \ (\mathcal{Z}_{\mathcal{C}}(G) \dashv \mathcal{Z}_{\mathcal{C}}(R))
$$

…since $\mathcal{Z}_{\mathcal{C}} \colon \mathcal{C}\text{-}\mathbf{BiMod} \to \mathbf{Cat}$ is a 2-functor

Functors on Drinfeld centers

We can now **compose**:

$$\mathcal{Z}(\mathcal{D}) \xrightarrow{\quad \mathcal{Z}(R) \quad} \mathcal{Z}_{\mathcal{C}}(\mathcal{C}) = \mathcal{Z}(\mathcal{C})$$

$$\mathcal{Z}(\mathcal{D}) \xrightarrow{F^G} \mathcal{Z}_{\mathcal{C}}(\mathcal{D}^G) \xrightarrow{\mathcal{Z}_{\mathcal{C}}(R)} \mathcal{Z}_{\mathcal{C}}(\mathcal{C})$$

$$F^G \colon \mathcal{Z}(\mathcal{D}) \hookrightarrow \mathcal{Z}(\mathcal{D}^G), \qquad (M, c^M) \mapsto (M, c^M_{G(-)})$$

Theorem (Flake–L.–Posur)

For a monoidal adjunction $G \dashv R$ *satisfying the projection formula,* R *induces a braided lax monoidal functor* $\mathcal{Z}(R) \colon \mathcal{Z}(\mathcal{D}) \to \mathcal{Z}(\mathcal{C})$, $(X, c) \mapsto (RX, c^R)$,

$$c^R_A = \left(RX \otimes A \xrightarrow{\mathrm{rproj}_{X,A}} R(X \otimes GA) \xrightarrow{R(c_{GA})} R(GA \otimes X) \xrightarrow{(\mathrm{lproj}_{A,X})^{-1}} A \otimes RX \right).$$

$$\mathrm{lax}^{\mathcal{Z}(R)}_{(X,c),(Y,d)} = \mathrm{lax}^R_{X,Y} \qquad \mathrm{lax}^{\mathcal{Z}(R)}_0 = \mathrm{lax}^R_0$$

Functoriality: $\mathcal{C} \xrightarrow{G_1} \mathcal{D} \xrightarrow{G_2} \mathcal{E}$, $G_i \dashv R_i$, $i = 1, 2 \implies \mathcal{Z}(R_1 R_2) = \mathcal{Z}(R_1)\mathcal{Z}(R_2)$

Implication and Examples

Corollary (Application)

The functor $\mathcal{Z}(\mathcal{D}) \xrightarrow{\mathcal{Z}(R)} \mathcal{Z}(\mathcal{C})$ maps (commutative) monoids in $\mathcal{Z}(\mathcal{D})$ to (commutative) monoids in $\mathcal{Z}(\mathcal{C})$.

Example:

- $\mathsf{H} \subset \mathsf{G}$ finite groups, monoidal adjunction

$$\mathrm{Rep}(\mathsf{G}) \underset{\mathrm{CoInd} \simeq \mathrm{Ind}}{\overset{\mathrm{Res}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Res}(\mathsf{H})$$

- $\mathcal{Z}(\mathrm{Rep}\,\mathsf{H}) \simeq {}^{\mathsf{H}}_{\mathsf{H}}\mathbf{YD}$ — Yetter–Drinfeld modules
  Objects: $V \in \mathrm{Rep}\,\mathsf{H}$ with coaction $\delta\colon V \to \mathsf{H} \otimes V$, $v \mapsto |v| \otimes v$, satisfying $|h \cdot v| = h|v|h^{-1}$
- Obtain braided lax monoidal functor $\mathcal{Z}(R)\colon {}^{\mathsf{H}}_{\mathsf{H}}\mathbf{YD} \to {}^{\mathsf{G}}_{\mathsf{G}}\mathbf{YD}$,
  $\mathcal{Z}(R)(V) = \mathsf{G} \otimes_{\mathsf{H}} V$ with coaction $\delta^{\mathrm{Ind}}(g \otimes v) = g|v|g^{-1} \otimes (g \otimes v)$

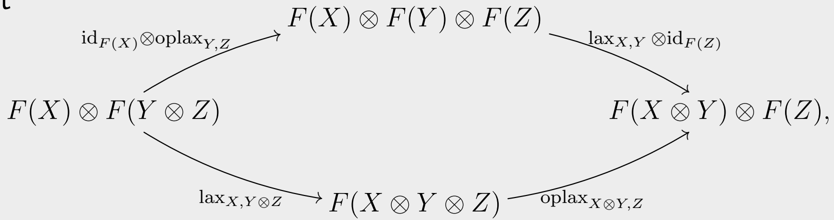
Frobenius monoidal functors

Definition

A Frobenius monoidal functor $F\colon \mathcal{D} \to \mathcal{C}$ is a lax and *oplax* monoidal functor

$$\mathrm{lax}_{X,Y}\colon F(X)\otimes F(Y) \longrightarrow F(X\otimes Y), \quad \mathrm{lax}_0\colon \mathbb{1} \longrightarrow F(\mathbb{1}),$$
$$\mathrm{oplax}_{X,Y}\colon F(X\otimes Y) \longrightarrow F(X)\otimes F(Y), \quad \mathrm{oplax}_0\colon F(\mathbb{1}) \longrightarrow \mathbb{1},$$

such that



and an analogous diagram, commute for any objects X, Y, Z of $\mathcal{D}$.

Example Any strong monoidal functor is Frobenius monoidal.

Ambiadjunctions

Definition

An *ambiadjunction* $F \dashv G \dashv F$ consists of:

- Functors $\mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{F} \mathcal{C}$,
- natural transformations
$$\mathrm{unit}^L \colon \mathrm{id}_{\mathcal{D}} \to GF, \quad \mathrm{counit}^L \colon FG \to \mathrm{id}_{\mathcal{C}}$$
which make F a *left adjoint* to G, $F \dashv G$,
- natural transformations
$$\mathrm{unit}^R \colon \mathrm{id}_{\mathcal{C}} \to FG, \quad \mathrm{counit}^R \colon GF \to \mathrm{id}_{\mathcal{D}},$$
which make F a *right adjoint* to G, $G \dashv F$.

- The functors F, G in an ambiadjunction is also called *Frobenius functors*.

Question: If G is *strong monoidal*, when is F or $\mathcal{Z}(F)$ *Frobenius monoidal*?

First examples

- Let H $\subset$ G be an inclusion of finite groups and consider the *strong monoidal* functor $\mathrm{Res}\colon \mathrm{Rep}\,\mathsf{G} \to \mathrm{Rep}\,\mathsf{H}$. Its left and right adjoints Ind and CoInd are isomorphic and we obtain an ambiadjunction $\mathrm{Ind} \dashv \mathrm{Res} \dashv \mathrm{Ind}$.
- For H a finite-dimensional Hopf algebra, the forgetful functor

$$G\colon H\text{-}\mathbf{Mod} \to \mathbf{Vect}$$

is *strong monoidal*. A non-zero right integral $\lambda\colon H \to \Bbbk$ for H^* gives an isomorphism $\mathrm{Ind} \cong \mathrm{CoInd}$.
- Let $\Bbbk\mathsf{C}_\ell = \Bbbk\left\langle g | g^\ell = 1 \right\rangle$ be the group algebra of a cyclic group of order ℓ and

$$T := \Bbbk\langle x, g | x^\ell = 0, g^\ell = 1, gx = \epsilon xg\rangle,$$

for $\epsilon \in \Bbbk^\times$ a primitive ℓ-th root of unity, the Taft algebra. It can be shown that Ind and CoInd are *non-isomorphic* for the inclusion $\Bbbk\mathsf{C}_\ell \hookrightarrow T$.

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Frobenius $\Rightarrow$ Frobenius monoidal

Recall that both adjunctions $F \dashv G$ and $G \dashv F$ come with a *right projection formula morphism*, rproj^R respectively rproj^L.

Theorem (F.–L.–P.)

Assume given an ambiadjunction $F \dashv G \dashv F$ with G strong monoidal. If

$$FX \otimes A \xrightarrow{\mathrm{rproj}^R_{A,X}} F(X \otimes GA) \quad \textit{and} \quad F(X \otimes GA) \xrightarrow{\mathrm{rproj}^L_{X,A}} FX \otimes A$$

are mutual inverses, then $F\colon \mathcal{D} \to \mathcal{C}$ with lax^F and oplax^F is a Frobenius monoidal functor.

Proof sketch:

- Assumptions $\iff F \dashv G \dashv F$ lifts to an ambiadjunction of right $\mathcal{C}$-module categories between

$$G\colon \mathcal{C} \to \mathcal{D}^G \text{ and } F\colon \mathcal{D}^G \to \mathcal{C}$$

Frobenius $\Rightarrow$ Frobenius monoidal

Proof sketch (continued):

- Composition with F, G induces functors

$$\mathrm{End}_{\mathbf{Mod}\text{-}\mathcal{C}}(\mathcal{C}) \underset{F\circ(-)\circ G}{\overset{G\circ(-)\circ F}{\rightleftarrows}} \mathrm{End}_{\mathbf{Mod}\text{-}\mathcal{C}}(\mathcal{D}^G)$$

- The ambiadjunction $F \dashv G \dashv F$ makes both compositions Frobenius monoidal functors
- There is a strong monoidal functor

$$\mathrm{Emb}\colon \mathcal{D} \to \mathrm{End}_{\mathbf{Mod}\text{-}\mathcal{C}}(\mathcal{D}^G), \quad X \mapsto X \otimes (-).$$

- There is an equivalence of monoidal categories

$$\mathcal{C} \xrightarrow{\sim} \mathrm{End}_{\mathbf{Mod}\text{-}\mathcal{C}}(\mathcal{C}), \quad X \mapsto X \otimes (-).$$

- The composition

$$\mathcal{D} \xrightarrow{\mathrm{Emb}} \mathrm{End}_{\mathbf{Mod}\text{-}\mathcal{C}}(\mathcal{D}^G) \xrightarrow{F\circ(-)\circ G} \mathrm{End}_{\mathbf{Mod}\text{-}\mathcal{C}}(\mathcal{C}) \simeq \mathcal{C}$$

  is isomorphic to F as both lax and oplax monoidal functor.
- Hence, F is Frobenius monoidal.

Lifting to the center

Both adjunctions $F \dashv G$ and $G \dashv F$ also have a *left projection formula morphism*, lproj^R respectively lproj^L.

Theorem (F.–L.–P.)

Assume given an ambiadjunction $F \dashv G \dashv F$ with G strong monoidal. If

$$\mathrm{rproj}^R_{A,X} = (\mathrm{rproj}^L_{X,A})^{-1} \text{ and } \mathrm{lproj}^R_{X,A} = (\mathrm{lproj}^L_{A,X})^{-1}$$

are mutual inverses, then $\mathcal{Z}(F) \colon \mathcal{Z}(\mathcal{D}) \to \mathcal{Z}(\mathcal{C})$ is a braided Frobenius monoidal functor.

- $\mathcal{Z}(F)$ has the *same* lax and oplax monoidal structures as F
- **Proof sketch:** Assumptions $\iff$ $F \dashv G \dashv F$ lifts to an ambiadjunction of $\mathcal{C}$-bimodule categories between

$$G \colon \mathcal{C} \to \mathcal{D}^G \text{ and } F \colon \mathcal{D}^G \to \mathcal{C}$$

Lifting to the center

Proof sketch (continued):

- If the projection formulas hold for the monoidal adjunction $G \dashv R$, then $\mathcal{Z}(R)$ is a braided lax monoidal functor.
- Dually, if the projection formulas hold for the *op*monoidal adjunction $L \dashv G$, then $\mathcal{Z}(L)$ is a braided *op*lax monoidal functor.
- The functors $\mathcal{Z}(R)$ and $\mathcal{Z}(L)$ are *different*, in general, even when $R = L$ as functors.
- The half braidings are different:

$$\mathcal{Z}(R)(X,c) = (R(X), c^{RX}_A = \left(RX \otimes A \xrightarrow{\mathrm{rproj}^R_{A,X}} R(X \otimes GA) \xrightarrow{R(c^X_{GA})} R(GA \otimes X) \xrightarrow{(\mathrm{lproj}^R_{X,A})^{-1}} A \otimes RX \right)$$

$$\mathcal{Z}(L)(X,c) = (L(X), c^{LX}_A = \left(LX \otimes A \xrightarrow{(\mathrm{rproj}^L_{A,X})^{-1}} L(X \otimes GA) \xrightarrow{L(c^X_{GA})} L(GA \otimes X) \xrightarrow{\mathrm{lproj}^L_{X,A}} A \otimes LX \right)$$

- Thus, for $F = R = L$, $\mathcal{Z}(R)$ and $\mathcal{Z}(L)$ *coincide* when

$$\mathrm{rproj}^R_{A,X} = (\mathrm{rproj}^L_{X,A})^{-1} \text{ and } \mathrm{lproj}^R_{X,A} = (\mathrm{lproj}^L_{A,X})^{-1}.$$

Hopf algebra examples

- $\varphi\colon K \hookrightarrow H$ an inclusion of Hopf algebras:
- Adjunctions: $$H\text{-}\mathbf{Mod} \underset{\mathrm{CoInd}}{\overset{\mathrm{Res}}{\rightleftarrows}} K\text{-}\mathbf{Mod}\ (\perp), \qquad H\text{-}\mathbf{Mod} \underset{\mathrm{Ind}}{\overset{\mathrm{Res}}{\rightleftarrows}} K\text{-}\mathbf{Mod}\ (\top)$$
- The projection formula *always* hold for Ind. If H is *finitely-generated projective* as a K-module, then the projection formulas hold for CoInd.
- $K \subset H$ is a Frobenius extension if there exists a Frobenius morphism $\mathrm{tr}\colon H \to K$ s.t. $H \cong \mathrm{Hom}_K(H, K) = \mathrm{CoInd}(K), 1 \mapsto \mathrm{tr}$, see e.g. [Fischmann–Montgomery–Schneider '97].
- If $K \subset H$ is a Frobenius extension then $\mathrm{Ind} \cong \mathrm{CoInd}$ and we have an ambiadjunction $\mathrm{Ind} \dashv \mathrm{Res} \dashv \mathrm{Ind}$.

Hopf algebra extensions

Theorem (F.–L.–P.)

If $H \subset K$ is a Frobenius extension of Hopf algebras such the Frobenius morphism $\mathrm{tr}\colon H \to K$ is a morphism of

(i) *right H-comodules*

(ii) *right and left H-comodules*

then

(i) $F\colon K\text{-}\mathbf{Mod} \to H\text{-}\mathbf{Mod}$ *is a Frobenius monoidal functor*

(ii) $\mathcal{Z}(F)\colon \mathcal{Z}(K\text{-}\mathbf{Mod}) \to \mathcal{Z}(H\text{-}\mathbf{Mod})$ *is a braided Frobenius monoidal functor.*

- (i) holds for *all* Frobenius extensions we know.
- (ii) needs *relative unimodularity*, e.g. *semisimplicity* of H.

Hopf algebra extensions

- **Recall:** $\mathcal{Z}(H\text{-}\mathbf{Mod}) \simeq {}^H_H\mathbf{YD}$ — Yetter–Drinfeld modules over H.
- Objects: H-modules V with a coaction $\delta^V(v) = v^{(-1)} \otimes v^{(0)}$ such that
$$\delta^V(h \cdot v) = h_{(1)} v^{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v^{(0)},$$
where $\Delta(h) = h_{(1)} \otimes h_{(2)}$ is the coproduct.
- The functor $\mathcal{Z}(F)$ is given by
$$\mathcal{Z}(F)(V, \delta^V) = \big(FV = \mathrm{Ind}(V) = H \otimes_K V,\ \delta^{FV}\big),$$
$$\delta^{FV}(h \otimes v) = h_{(1)} v^{(1)} S(h_{(3)}) \otimes (h_{(2)} \otimes v^{(0)}).$$
- Condition (i) holds for large classes of Frobenius extensions of Hopf algebras, (ii) is more special. First examples:
  - For $\Bbbk$H $\subset$ $\Bbbk$G group algebras, (ii) holds.
  - For $\Bbbk \subset H$, H finite-dimensional, (i) holds. (ii) is equivalent to H^* being unimodular.

Examples

- Consider the small quantum group $u_\epsilon(\mathfrak{sl}_2)$ for ϵ a primitive ℓ-th root of unity ϵ. The Cartan part is the group algebra $\Bbbk\mathsf{C}_\ell$. The extension $\Bbbk\mathsf{C}_\ell \subset u_\epsilon(\mathfrak{sl}_2)$ satisfies (i) but not (ii). Hence
$$\mathrm{Ind}\colon \Bbbk\mathsf{C}_\ell\text{-}\mathbf{Mod} \to u_\epsilon(\mathfrak{sl}_2)\text{-}\mathbf{Mod}$$
is a Frobenius monoidal functor but does *not* extend to Drinfeld centers.
- The (Kac–De Concini) quantum group $U_\epsilon(\mathfrak{g})$ contains a large commutative Hopf subalgebra $Z = \Bbbk[E_i^\ell, F_i^\ell, K_i^{\pm\ell}]$, the algebra of functions $\mathcal{O}_{\mathsf{H}}$ of an algebraic group H. The inclusion $Z \subset U_\epsilon(\mathfrak{g})$ satisfies (i) but not (ii). $\Rightarrow$ Frobenius monoidal functor
$$\mathrm{Ind}\colon \mathbf{QCoh}(\mathsf{H}/^{\mathrm{ad}}\mathsf{H}) \to U_\epsilon(\mathfrak{g})\text{-}\mathbf{Mod}$$
- In both cases, we still have *lax* and *oplax* monoidal functors on the center.
- If H is a finite-dimensional semisimple and co-semisimple Hopf algebra , then any extension of Hopf algebras $K \subset H$ satisfies (ii).

. . . Thank you for your attention!