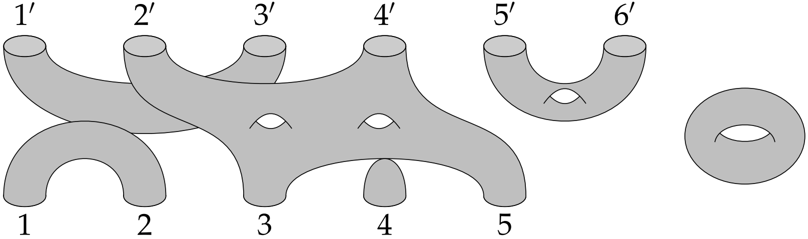
Indecomposable objects in Khovanov–Sazdanovic cobordism categories and rings of modular symmetric functions



ROBERT LAUGWITZ – University of Nottingham

UEA Pure Maths Seminar — November 22, 2022

SUMMARY

- Khovanov–Sazdanovic cobordism categories DCob_α, $\alpha = p(x)/q(x)$ a rational function
- Special case: Deligne's category $\underline{\mathrm{Rep}}_{\Bbbk}(S_t)$ interpolating $\mathrm{Rep}(S_n)$ to limits $t \in \Bbbk$
- $\operatorname{gr} \mathrm{K}_0(\mathrm{DCob}_\alpha) \cong \bigotimes_{z \in Z} \mathrm{Sym}$
  Z: zero set of a polynomial associated to α
- Techniques developed:
  - *associated graded categories* of Krull–Schmidt categories
  - *Galois descent* for categorical group actions
- Works in $\operatorname{char} \Bbbk = p$, giving rings of *modular symmetric functions*
- Joint work with Johannes Flake (Aachen) & Sebastian Posur (Münster) `ArXiv:2106.05798`

CONTENTS

Deligne's Interpolation Category $\underline{\mathrm{Rep}}(S_t)$

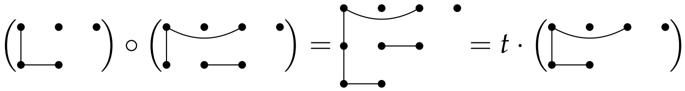
Motivation: Let $V := \mathbb{C}^n$ *standard representation* of S_n.

- Every simple S_n-representation is a direct summand of $V^{\otimes k}$ for some $k \geq 0$.
- Partitions of $\{1, \ldots, k, 1', \ldots, l'\}$ give morphisms of S_n-representations
$$V^{\otimes k} \to V^{\otimes l}$$
- These morphisms span $\mathrm{Hom}_{S_n}(V^{\otimes k}, V^{\otimes l})$ as a $\Bbbk$-vector space.
- $\mathrm{Rep}(S_n)$ is the *idempotent completion* (the *Karoubian envelope*) of the full tensor subcategory generated by V.
- Deligne: Composition rule is combinatorial, the number n appears polynomially.
- replacing n by $t \in \mathbb{C}$ gives new tensor categories $\underline{\mathrm{Rep}}(S_t)$

DELIGNE'S INTERPOLATION CATEGORY

$\underline{\mathrm{Rep}}(S_t) = \underline{\mathrm{Rep}}_{\Bbbk}(S_t)$ is the *idempotent completion* of the following category:

- Objects: $[m]$ for $m \in \mathbb{Z}_{\geq 0}$
- Morphisms $[m] \to [k]$: Linear combinations of *partitions* of $\{1, \ldots, m, 1', \ldots, k'\}$
- Composition: Concatenation — for example,



Deligne '07: Symmetric tensor category $\underline{\mathrm{Rep}}(S_t)$ for $t \in \Bbbk$

- For *generic* $t \notin \mathbb{Z}_{\geq 0}$: $\underline{\mathrm{Rep}}(S_t)$ is semisimple
- For $n \in \mathbb{N}$, there is a symmetric tensor functor

$$\mathcal{F}_n : \quad \underbrace{\underline{\mathrm{Rep}}(S_n)}_{\text{not semisimple}} \longrightarrow\!\!\!\!\!\rightarrow \underbrace{\mathrm{Rep}(S_n)}_{\text{semisimplification}}$$

which is full & essentially surjective

Indecomposable objects in $\underline{\operatorname{Rep}}_{\mathbb{C}}(S_t)$

Theorem (Deligne for $t \notin \mathbb{Z}_{\geq 0}$, Comes–Ostrik for t general)

There is an isomorphism of graded rings

$$\operatorname{gr} \mathrm{K}_0 \, \underline{\operatorname{Rep}}_{\mathbb{C}}(S_t) \cong \operatorname{Sym}$$

Associated graded ring

Grothendieck ring

In particular, indecomposable objects correspond to Young diagrams.

Here, $\operatorname{Sym} = \bigoplus_{n \geq 0} \mathrm{K}_0 \operatorname{Rep}_{\mathcal{C}}(S_n)$ is the *ring of symmetric functions* with induction product

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu = \left[\operatorname{Ind}^{S_{n+m}}_{S_n \times S_m} (V_\lambda \boxtimes V_\mu) \right],$$

where $s_\lambda = [V_\lambda]$ is the Schur function of $\lambda \vdash n$, and $\mu \vdash m$, and $c_{\lambda,\mu}$ Littlewood–Richardson coefficients.

Krull–Schmidt categories

Definition

An additive category $\mathcal{C}$ is *Krull–Schmidt* if every object X has a decomposition $X = X_1 \oplus \ldots \oplus X_n$, where $\mathrm{End}_{\mathcal{C}}(X_i)$ is local for all i.

- $\mathcal{C}$ is Karoubian (additive and idempotent complete)
- X is *indecomposable* $\iff \mathrm{End}_{\mathcal{C}}(X)$ is local
- Decompositions into indecomposables is unique up to permulation of summands—the Krull–Schmidt theorem
- Examples:
  - Representation categories of groups
  - If $\mathcal{C}$ is Karoubian, $\Bbbk$-linear, with all $\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(X, Y) < \infty$, then $\mathcal{C}$ is Krull–Schmidt
  - $\Longrightarrow \underline{\mathrm{Rep}}_{\Bbbk}(S_t)$ is Krull–Schmidt

CLASSIFYING INDECOMPOSABLES

Let $\mathcal{C}$ be a Krull–Schmidt category

- Take an object $X \neq 0$ in $\mathcal{C}$
- $\left\{\begin{matrix}\text{indecomposable}\\ \text{summands in } X\end{matrix}\right\} /_{\text{iso}} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{primitive}\\ \text{idempotents}\\ \text{in } \mathrm{End}_{\mathcal{C}}(X)\end{matrix}\right\} \Big/ \text{conjugation}$
- Pass to *quotient category* $\mathcal{C}/\langle X\rangle$
  - $\mathcal{C}/\langle X\rangle$ formally sets $\mathrm{Id}_X = 0$
  - The quotient is still Krull–Schmidt
- Continue with $\mathcal{C}/\langle X\rangle$ using that

$$\left\{\begin{matrix}\text{indecomposables}\\ \text{in } \mathcal{C}\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{indecomposables}\\ \text{in } \mathcal{C}/\langle X\rangle\end{matrix}\right\} \sqcup \left\{\begin{matrix}\text{indecomposables}\\ \text{in } X\end{matrix}\right\}$$

The example $\underline{\mathrm{Rep}}_{\Bbbk}(S_t)$

Let $\Bbbk$ be any field with $\operatorname{char} \Bbbk = p$, consider $\mathcal{C} = \underline{\mathrm{Rep}}_{\Bbbk}(S_t)$.

- $[0]$ is indecomposable as $\mathrm{End}_{\mathcal{C}}([0]) = \Bbbk$
- In $\mathcal{C}/\langle [0] \rangle$, $[1]$ is indecomposable as $\mathrm{End}([1]) = \Bbbk$
- Inductively, consider $[n+1]$ in $\mathcal{C}/\langle [0], \dots, [n] \rangle$.
- One proves that in this quotient $\mathrm{End}([n+1]) \cong \Bbbk S_{n+1}$ is the *group algebra* of S_{n+1}
- Use the classical result

$$\left\{ \begin{matrix} \text{primitive} \\ \text{idempotents} \\ \text{in } \Bbbk S_{n+1} \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} p\text{-regular Young} \\ \text{diagrams of} \\ \text{size } n+1 \end{matrix} \right\}$$

- If $\operatorname{char} \Bbbk = 0$, p-regular condition superfluous

A FIRST RESULT

Let $\mathbb{k}$ be a field with $\operatorname{char}\mathbb{k} = p$ and $t \in \mathbb{k}$.

Theorem (Flake–L.–Posur)

There is an isomorphism of graded rings

$$\operatorname{gr} \mathrm{K}_0 \, \underline{\mathbf{Rep}}_{\mathbb{k}}(S_t) \cong \mathrm{Sym}^p.$$

Indecomposables correspond to p-regular Young diagrams.

The right hand side is

$$\mathrm{Sym}^p = \bigoplus_{n \geq 0} \mathrm{K}_0(\mathbb{k}S_n\text{–}\mathbf{proj}_{\mathbb{k}})$$

with multiplication induced by the *induction product*.
This is an analogue of Sym for $\operatorname{char}\mathbb{k} = p$ (independent of $\mathbb{k}$ besides dependence on p).

The ring Sym^p

We fix

- $\Bbbk$ a *field* of characteristic p
- R a *complete discrete valuation ring* of characteristic 0 such that $R/\operatorname{rad}(R) \cong \Bbbk$
- $\mathbb{K} := \operatorname{Quot}(R)$ the *quotient field*

The canonical ring maps $\Bbbk \leftarrow R \rightarrow \mathbb{K}$ induce functors

$$\Bbbk S_n\text{–}\mathbf{proj} \xleftarrow{(\Bbbk \otimes_R -)} RS_n\text{–}\mathbf{proj} \xhookrightarrow{(\mathbb{K} \otimes_R -)} \mathbb{K}S_n\text{–}\mathbf{mod}$$

These induce

$$\mathrm{K}_0(\Bbbk S_n\text{–}\mathbf{proj}) \xleftarrow[\text{isomorphism}]{\mathrm{K}_0(\Bbbk \otimes_R -)} \mathrm{K}_0(RS_n\text{–}\mathbf{proj}) \xrightarrow[\text{split mono}]{\mathrm{K}_0(\mathbb{K} \otimes_R -)} \mathrm{K}_0(\mathbb{K}S_n\text{–}\mathbf{mod})$$

Thus, we have an embedding of rings

$$\mathrm{Sym}^p \hookrightarrow \mathrm{Sym}.$$

Coefficients on the Schur basis: (decomposition matrix)^T

Example

We can write down the embedding $\mathrm{Sym}^2 \hookrightarrow \mathrm{Sym}$ in $\deg \leq 4$

$$[V_{(0)}] \mapsto [V_{(0)}], \qquad [V_{(1)}] \mapsto [V_{(1)}], \qquad [V_{(2)}] \mapsto [V_{(2)}] + [V_{(1^2)}]$$
$$[V_{(2,1)}] \mapsto [V_{(2,1)}], \qquad [V_{(3)}] \mapsto [V_{(3)}] + [V_{(1^3)}]$$
$$[V_{(4)}] \mapsto [V_{(4)}] + [V_{(3,1)}] + [V_{(2,1^2)}] + [V_{(1^4)}]$$
$$[V_{(3,1)}] \mapsto [V_{(3,1)}] + [V_{(2^2)}] + [V_{(2,1^2)}]$$

and compute products in Sym^2

$$[V_{(1)}] \cdot [V_{(1)}] = [V_{(2)}] \qquad [V_{(1)}] \cdot [V_{(3)}] = [V_{(4)}]$$
$$[V_{(1)}] \cdot [V_{(2)}] = [V_{(3)}] + [V_{(2,1)}] \qquad [V_{(1)}] \cdot [V_{(2,1)}] = [V_{(3,1)}]$$
$$[V_{(2)}] \cdot [V_{(2)}] = [V_{(4)}] + 2[V_{(3,1)}]$$

THE ASSOCIATED GRADED CATEGORY

$\mathcal{C} = \underline{\mathrm{Rep}}_{\Bbbk}(S_t)$ has a natural *filtration*:

$$\mathcal{C}_n = \langle [0], \ldots, [n] \rangle.$$

The set $\{[n] | n \geq 0\}$ *generates* $\mathcal{C}$ and hence $\mathcal{C} = \coprod_{n \geq 0} \mathcal{C}_n$.

Use the subquotient Krull–Schmidt categories $\mathcal{D}_n := \mathcal{C}_n / \mathcal{C}_{n-1}$ to define the *associated graded category*

$$\mathrm{gr}\, \mathcal{C} = \bigoplus_{n \geq 0} \mathcal{D}_n.$$

The associated graded category

- is Krull–Schmidt
- inherits a tensor product from $\mathcal{C}$, since $\mathcal{C}_i \otimes \mathcal{C}_j \subseteq \mathcal{C}_{i+j}$
- does *not* have duals

THE ASSOCIATED GRADED CATEGORY

Proposition (Flake–L.–Posur)

If a filtration of a monoidal Krull–Schmidt category $\mathcal{C}$ satisfies $\mathcal{C}_i \otimes \mathcal{C}_j \subseteq \mathcal{C}_{i+j}$, then we have an isomorphism of graded rings

$$\mathrm{K}_0(\operatorname{gr} \mathcal{C}) \cong \operatorname{gr} \mathrm{K}_0(\mathcal{C}) \, .$$

The following equivalence "categorifies" $\operatorname{gr} \mathrm{K}_0 \, \underline{\mathrm{Rep}}(S_t) \cong \mathrm{Sym}^p$.

Example

For any field $\Bbbk$, there is an *equivalence* of $\Bbbk$-linear symmetric monoidal categories

$$\operatorname{gr} \underline{\mathrm{Rep}}_{\Bbbk}(S_t) \simeq \bigoplus_{n \geq 0} \Bbbk S_n\textbf{–proj},$$

which is compatible with gradings.

THE CATEGORY OF 2-COBORDISMS

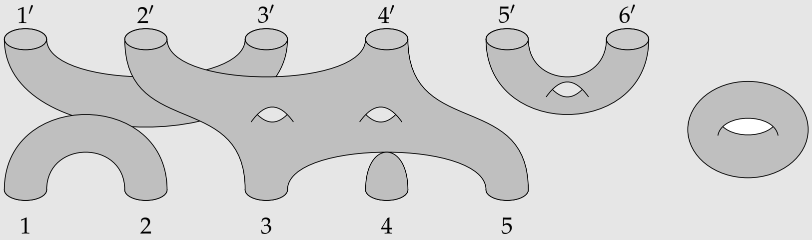
Cob_2 denotes the category of oriented *two-dimensional cobordisms*

- Objects $[n] = S^1 \sqcup \ldots \sqcup S^1$ correspond to n circles
- Morphisms $[m] \to [n]$ are oriented two-dimensional cobordism with boundary given by a m incoming and n outgoing boundary components
- $[n] \otimes [m] = [n+m]$ and disjoint union of cobordisms makes Cob_2 *symmetric monoidal*
- Linearize Cob_2 over a field $\Bbbk$ taking *formal linear combinations* of morphisms

A 2-COBORDISM

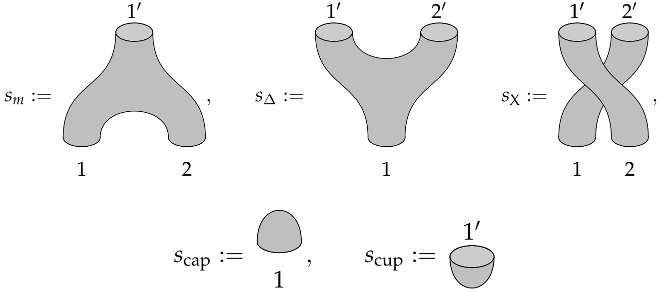
Example

A morphism $[5] \to [6]$ in Cob_2 is given by:



Generators for Cob_2

Define basic cobordisms



Lemma

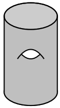
The monoidal category Cob_2 is generated under composition, tensor product, and $\mathbb{k}$-linear combinations by the morphisms

$s_m, s_\Delta, s_\chi, s_{\mathrm{cup}}, s_{\mathrm{cap}}$.

The tensor categories DCob_α

- Let $\alpha = p(x)/q(x)$ be a *rational series*, such that $q(0) = 1$, $\gcd(p(x), q(x)) = 1$
- For $k = \min\{\deg p(x) + 1, q(x)\}$ set $u_\alpha(x) = x^k q(x^{-1})$
- Consider the morphism $x\colon [1] \to [1]$ of genus 1,

$$x = s_\Delta \circ s_m =$$



Definition (Khovanov–Sazdanovic, 2020)

DCob_α is the *quotient tensor category* of DCob_2 by the relations

$$u_\alpha(x) = 0, \qquad s_g = \alpha_g \operatorname{Id}_{[0]},$$

where $s_g\colon [0] \to [0]$ is a *closed genus g surface*.

THE TENSOR CATEGORIES DCob_α

- DCob_α is a $\Bbbk$-linear *symmetric tensor category*
- DCob_α is a *Krull–Schmidt category*
- $\dim_{\Bbbk} \mathrm{Hom}_{\mathrm{DCob}_\alpha}(X, Y) < \infty$ for all objects (as α is rational)
- The object $[1]$ *generates* DCob_α under taking tensor products and Karoubian completion

Theorem (Khovanov–Kononov–Ostrik)

Let α be a rational function and choose a partial fraction decomposition

$$\alpha(t) = \sum_{i=1}^{l} \frac{p_i(t)}{q_i(t)} + \alpha_0(t), \qquad \alpha_0(t) \in \Bbbk[t], \quad \deg p_i(t) < \deg q_i(t),$$

where the $q_i(t)$ have a unique zero, set $\alpha_i := p_i(t)/q_i(t)$. Then there is an equivalence of Karoubian tensor categories

$$F_\alpha^D \colon \mathrm{DCob}_\alpha \xrightarrow{\sim} \boxtimes_{i=0}^{l} \mathrm{DCob}_{\alpha_i}, \quad [1] \mapsto \bigoplus_i [0]^{\boxtimes(i-1)} \boxtimes [1] \boxtimes [0]^{\boxtimes(n-i)}.$$

SPECIAL CASES OF DCob_α

Example

1. Let $\alpha = t/(1-x)$. Then $u_\alpha(x) = x - 1$ and hence $\mathrm{DCob}_\alpha \simeq \underline{\mathrm{Rep}}_{\Bbbk}(S_t)$.
2. Let $\alpha = \beta_0 + \beta_1 x$, $\beta_1 \neq 0$. Then $\mathrm{DCob}_\alpha \simeq \underline{\mathrm{Rep}}_{\Bbbk}(O_{\beta_1 - 2})$, Deligne's interpolation category of $\mathrm{Rep}(O_n)$.
3. The *semisimplification* of DCob_c, for $c \in \mathbb{C}^\times$, is the category $\mathrm{Rep}^+\mathfrak{osp}(1|2)$ ($\otimes$-generated by the standard module of $\mathfrak{osp}(1|2)$) by [KOK20].

- The theorem of [KOK20] tells us that over $\Bbbk$ a splitting field for $u_\alpha(x)$, DCob_α is an *external tensor product* of DCob_{α_i}, indexed by the *distinct roots* of $u_\alpha(x)$.
- [KOK20] also determine when *semisimplification* and *abelianization* of DCob_α exist. This is equivalent to $q(x)$ having non-repeating roots in $\overline{\Bbbk}$ and $\deg p(x) \leq \deg q(x) + 1$.

THE SEPARABLE CASE

Theorem (Flake–L.–Posur)

Assume that $\Bbbk$ is a splitting field for $u_\alpha(x)$. Then there is an equivalence of symmetric monoidal categories

$$\operatorname{gr}\mathrm{DCob}_\alpha \simeq \bigoplus_{n\geq 0}(P_n \rtimes \Bbbk S_n)\text{–}\mathbf{proj}.$$

- $P_n = \Bbbk[x_1,\ldots,x_n]/\big(u_\alpha(x_1),\ldots,u_\alpha(x_n)\big)$, where $x_i = \mathrm{Id}_{[i-1]} \otimes s_m s_\Delta \otimes \mathrm{Id}_{[n-i]}$
- $P_n \rtimes \Bbbk S_n \subset \mathrm{End}_{\mathrm{DCob}_\alpha}([n])$
- Consider the *filtration* on $\mathcal{C} = \mathrm{DCob}_\alpha$ given by $\mathcal{C}_n = \langle [0],\ldots,[n]\rangle$
- Then $\mathcal{C}_{n+1}/\mathcal{C}_n \simeq P_n \rtimes \Bbbk S_n$–$\mathbf{proj}$

GRADED GROTHENDIECK RING (SEPARABLE CASE)

For a polynomial $u(x) \in \Bbbk[x]$, denote

$$\mathcal{S}_{u(x)} := \bigoplus_{n \geq 0} P_n \rtimes \Bbbk S_n\text{–}\mathbf{proj},$$

for $P_n = \Bbbk[x_1, \ldots, x_n] / \big(u(x_1), \ldots, u(x_1)\big)$.

Corollary (Flake–L.–Posur)

Let $u'_\alpha(x)$ feature all irreducible factors of $u_\alpha(x)$ precisely once. Then

$$\operatorname{gr} \mathrm{K}_0(\mathrm{DCob}_\alpha) = \mathrm{K}_0\left(\mathcal{S}_{u_\alpha(x)}\right) = \mathrm{K}_0\left(\mathcal{S}_{u'_\alpha(x)}\right).$$

The proof uses an equivalence of symmetric monoidal categories $\mathcal{S}_{u'_\alpha(x)} \simeq \mathcal{S}_{u_\alpha(x)}/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by all radical morphisms of P_n which is contained in the radical of the category $\mathcal{S}_{u_\alpha(x)}$.

The general case

Let $\alpha \in \Bbbk[[x]]$ be a rational series, $\operatorname{char} \Bbbk = p \geq 0$.

- $\mathbb{K}|\Bbbk$ be the *splitting field* of $u_\alpha(x)$
- $G := \operatorname{Aut}(\mathbb{K}|\Bbbk)$ the *Galois group*
- G acts on the *zero set* $Z := \{z \in \mathbb{K} | u_\alpha(z) = 0\}$

Theorem (Flake–L.–Posur)

There is an isomorphism of graded rings

$$\operatorname{gr} \mathrm{K}_0(\mathrm{DCob}_\alpha) \cong \left(\bigotimes\nolimits_{z\in Z} \mathrm{Sym}^p\right)^G$$

- $\mathrm{K}_0(\mathrm{DCob}_\alpha) \cong \mathrm{K}_0\left(\mathcal{S}_{u'_\alpha(x)}\right)$ factors into a tensor product of copies of Sym^p using [KOK20]
- G-invariants are taken w.r.t. G permuting the tensor factors Sym^p according to $G \curvearrowright Z$

EXAMPLE OF GENERAL RESULT

Example

Consider $\alpha(t) = \frac{c_0+c_1 t}{1-\beta_1 t+\beta_0 t^2} \in \mathbb{R}[[t]]$ with *irreducible denominator*

- Then $u_\alpha(t) = \beta_0 - \beta_1 t + t^2$
- *Splitting field*: $\mathbb{C}|\mathbb{R}$, *Galois group*: $G = \mathrm{Gal}(\mathbb{C}|\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z} = \langle \sigma | \sigma^2 = 1 \rangle$
- $u_{\alpha^{\mathbb{K}}}(t) = (t-\rho)(t-\overline{\rho})$, for $\rho \in \mathbb{C} \setminus \mathbb{R}$, $\sigma\rho = \overline{\rho}$

The theorem implies

$$\mathrm{gr}\, \mathrm{K}_0(\mathrm{DCob}_\alpha) \cong (\mathrm{Sym} \otimes \mathrm{Sym})^{\mathbb{Z}/2\mathbb{Z}},$$

consisting of

$$[V_\lambda] \otimes [V_\lambda], \qquad [V_{\lambda_1}] \otimes [V_{\lambda_2}] + [V_{\lambda_2}] \otimes [V_{\lambda_1}], \quad \text{for } \lambda_1 \neq \lambda_2$$

SKETCH OF THE PROOF

A categorical statement

Proposition

For $G = \mathrm{Gal}(\mathbb{K}|\Bbbk)$, there is an equivalence of symmetric monoidal categories

$$\mathrm{gr}\,\mathrm{DCob}_\alpha \simeq {\mathcal{S}_{u_\alpha^{\mathbb{K}}(x)}}^G,$$

compatible with gradings, where $u_\alpha^{\mathbb{K}}(x)$ is $u_\alpha(x)$ viewed in $\mathbb{K}[x]$.

For a strict categorical G-action

$$T\colon G \to \mathrm{End}(\mathcal{C}), \quad g \mapsto T_g,$$

the G-equivariantization $\mathcal{C}^G$ consists of objects

$$X \in \mathcal{C}, \qquad \text{with isomorphisms} \quad e_g\colon T_g(X) \to X$$
$$\alpha_1 = \mathrm{Id}_X, \qquad e_h \circ T_h(e_g) = e_{hg}, \quad \forall g, h \in G.$$

Sketch of the proof

The proposition follows from a categorical *Galois descent*

Theorem (Galois descent)

Given a finite Galois extension $G = \mathrm{Gal}(\mathbb{K}|\Bbbk)$, $\mathcal{C}$ *a* $\Bbbk$*-linear hom-finite Krull–Schmidt category,* $\mathcal{C}^{\mathbb{K}} := \mathrm{Kar}(\mathcal{C} \boxtimes_{\Bbbk} \mathbf{vect}_{\mathbb{K}})$*, there is an equivalence*

$$\mathcal{C} \xrightarrow{\sim} (\mathcal{C}^{\mathbb{K}})^G \colon X \mapsto (X^{\mathbb{K}}, (\mathrm{Id}_{X^{\mathbb{K}}})_g).$$

Apply the result to $\mathcal{C} = \mathcal{S}_{u_\alpha(x)}$, $u_\alpha(x) \in \Bbbk[x]$

$$\implies \qquad \mathcal{C}^{\mathbb{K}} \cong \mathcal{S}_{u_\alpha^{\mathbb{K}}(x)},$$

with $u_\alpha^{\mathbb{K}}(x) = u_\alpha(x) \in \mathbb{K}[x]$

SKETCH OF THE PROOF

In general, the *equivalence* $\mathcal{C} \simeq (\mathcal{C}^{\mathbb{K}})^G$ only gives an *inclusion*

$$\mathrm{K}_0(\mathcal{C}) \hookrightarrow \mathrm{K}_0(\mathcal{C}^{\mathbb{K}})^G.$$

Example

$\mathcal{C} := \mathbb{H}$–**mod** is an $\mathbb{R}$-linear category viewing $\mathbb{H}$ as an $\mathbb{R}$-algebra. $\mathcal{C}^{\mathbb{C}} \simeq \mathrm{Mat}_2(\mathbb{C})$–**mod** $\simeq \mathbb{C}$–**mod** as $\mathbb{C}$-linear categories, since $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Mat}_2(\mathbb{C})$. We obtain:

$$\mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}^{\mathbb{C}})^G, \quad [\mathbb{H}] \mapsto 2[\mathbb{C}] \subsetneq \langle[\mathbb{C}]\rangle$$

To prove that $\mathrm{K}_0(\mathcal{S}^G_{u_\alpha(x)}) \cong \mathrm{K}_0({\mathcal{S}_{u_\alpha(x)}}^G)$ we construct *G-equivariant* objects categorifying the *orbit sums* in $\mathrm{K}_0({\mathcal{S}_{u'_\alpha(x)}}^G)$

CONCLUDING COMMENTS

- There are other Deligne interpolation categories, $\underline{\mathrm{Rep}}(GL_t)$, $\underline{\mathrm{Rep}}(O_t)$, wreath products, ...
- F. Knop and recently Ehud Meir defined a framework for general constructions of interpolation categories
- Classification questions of indecomposables can be posted for such categories
- Our techniques can be used working over general fields $\Bbbk$ of coefficients

Thank you very much for your attention!

SOME REFERENCES

[CO11] J. Comes and V. Ostrik, *On blocks of Deligne's category* $\underline{\mathrm{Rep}}(S_t)$, Adv. Math. **226** (2011), no. 2, 1331–1377.

[Del07] P. Deligne, *La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel*, Algebraic groups and homogeneous spaces, 2007, pp. 209–273.

[FLP21] J. Flake, R. Laugwitz, and S. Posur, *Indecomposable objects in Khovanov-Sazdanovic's generalizations of Deligne's interpolation categories*, arXiv e-prints (2021), arXiv:2106.05798.

[KOK20] M. Khovanov, V. Ostrik, and Y. Kononov, *Two-dimensional topological theories, rational functions and their tensor envelopes*, arXiv e-prints (November 2020), arXiv:2011.14758.

[KS20] M. Khovanov and R. Sazdanovic, *Bilinear pairings on two-dimensional cobordisms and generalizations of the Deligne category*, arXiv e-prints (July 2020), arXiv:2007.11640.

[Mei21] E. Meir, *Interpolations of monoidal categories and algebraic structures by invariant theory*, arXiv e-prints (2021), arXiv:2105.04622.