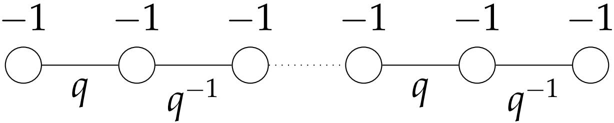
Relative Drinfeld centers and non-semisimple modular tensor categories



ROBERT LAUGWITZ — UNIVERSITY OF NOTTINGHAM

Universal Quantum Symmetries Lectures (UQSL)
November 8, 2022

Summary

- We construct non-semisimple *modular tensor categories* $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$
  - $\mathcal{B}$ is a braided category, e.g. $\mathcal{B} = H$–**mod**
    — H a quasi-triangular Hopf algebra
  - $\mathcal{C}$ is a finite tensor category with a *central functor* $\mathcal{B}^{\text{rev}} \to \mathcal{C}$
  - $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ is the *relative center* of $\mathcal{C}$
- Examples:
  - For B a Hopf algebra object in H–**mod**,

$$\mathcal{Z}_{\mathcal{B}}(H\text{–}\mathbf{mod}(\mathcal{B})) \simeq \mathrm{Drin}_H(B)\text{–}\mathbf{mod},$$

    where $\mathrm{Drin}_H(B)$ is the *relative Drinfeld double*
  - For A commutative algebra in $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$,

$$\mathcal{Z}_{\mathcal{B}}(\mathrm{Rep}_{\mathcal{C}}(A)) \cong \mathrm{Rep}^{\mathrm{loc}}_{\mathcal{Z}_{\mathcal{B}}(\mathcal{C})}(A),$$

    the category of *local A-modules*.
- Joint work with C. Walton (Rice) `ArXiv:2010.11872`, `ArXiv:2202.08644`
- Joint work in progress with G. Sanmarco (Iowa), and A. Ros Camacho, S. Hannah (Cardiff)

CONTENTS

MOTIVATION

Modular tensor categories

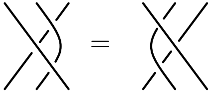
- ...are algebraic structure used in the construction of the Reshetikhin–Turaev 3D *topological field theory*
- ...capture the structure of 2D *conformal field theory*
- ...are are typically *semisimple*
- Recent advances, *non-semisimple* modular categories:
  - Equivalent characterizations of modularity [Shi19]
  - Non-semisimple 3-manifold invariants [DRGG+19], ...
  - Modular functor valued in chain complexes [SW21]
- Key example: $\mathrm{u}_q(\mathfrak{g})$–**mod**, $q^N = 1$ with N odd, is *non-semisimple* modular (Reshetikhin–Turaev take a *semisimple quotient category*)
- Other examples: Lentner–Ohrmann (2016), Gainutdinov–Lentner–Ohrmann (2018), Negron (2018)
- Our goal: Construction of non-semisimple modular categories using representation theory

Modular Categories

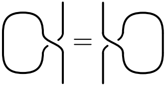
A modular category $\mathcal{C}$ is:

- a $\Bbbk$-linear[1] *finite abelian category*
  — finite dimensional Hom spaces, finitely many simple objects
- a *tensor category*
  — tensor product $\otimes$, unit $\mathbb{1}$, $\mathrm{End}(\mathbb{1}) = \Bbbk$, left and right *duals* ${}^*V, V^*$
- equipped with a *braiding* $\Psi_{V,W} \colon V \otimes W \xrightarrow{\sim} W \otimes V$

$$\begin{aligned} \Psi_{V,W\otimes U} &= (\mathrm{Id} \otimes \Psi_{V,U})(\Psi_{V,W} \otimes \mathrm{Id}) \\ \Psi_{V\otimes W,U} &= (\Psi_{V,U} \otimes \mathrm{Id})(\mathrm{Id} \otimes \Psi_{W,U}) \end{aligned} \implies$$



- a *ribbon category* — natural isomorphisms ${}^*V \xrightarrow{\sim} V^*$ s.t. left and right *twists* coincide



- *non-degenerate* — $(\forall W \in \mathcal{C} : \Psi_{W,V}\Psi_{V,W} = \mathrm{Id}) \implies V \cong \mathbb{1}^{\oplus k}$

[1] Assume $\Bbbk = \overline{\Bbbk}$ throughout

The Monoidal/Drinfeld Center $\mathcal{Z}(\mathcal{C})$

$\mathcal{Z}(\mathcal{C})$ consists of:

- **Objects:** (V,c), $V \in \mathcal{C}$, *half-braiding* $c_W \colon V \otimes W \to W \otimes V$, natural in W, s.t.

$$c_{W\otimes U} = (\mathrm{Id}_W \otimes c_U)(c_W \otimes \mathrm{Id}_U)$$

$$\Longrightarrow \quad \Psi_{V,W} = c_{V,W} \quad \text{gives a braiding}$$

- **Morphisms:** morphisms in $\mathcal{C}$ which commute with the half-braidings

Theorem (Drinfeld, Majid, Joyal–Street ∼1990)

For $\mathcal{C}$ a tensor category, $\mathcal{Z}(\mathcal{C})$ is a braided tensor category

EXAMPLES

$\mathcal{C} = H$–**mod** — H (finite-dimensional) $\Bbbk$-Hopf algebra
$\Longrightarrow \mathcal{C}$ is a tensor category, with $\otimes$ via *coproduct* $\Delta\colon H \to H \otimes_{\Bbbk} H$

Question: What is the center $\mathcal{Z}(\mathcal{C})$ in this case?

Answer: Modules over the Drinfeld double $\mathrm{Drin}(H)$

$\mathrm{Drin}(H) \stackrel{\Bbbk}{=} H \otimes_{\Bbbk} H^*$ with H, H^* Hopf subalgebras.

Example ((Twisted) group case)

Take $H = \Bbbk G$ a group algebra. Then $\mathrm{Drin}(G) \stackrel{\Bbbk}{=} \Bbbk G \otimes \Bbbk[G]$,

$$g\delta_h = \delta_{ghg^{-1}}g, \qquad \forall g, h \in G.$$

More generally, include a 3-cocycle ω on the group: $\mathrm{Drin}^{\omega}(G)$ and $\mathrm{Drin}^{\omega}(G)$–**mod** $\simeq \mathcal{Z}(\mathbf{vect}_G^{\omega})$

$\mathrm{Drin}^{\omega}(G)$–**mod** are *modular categories*, semisimple if $\mathrm{char}\,\Bbbk = 0$.

QUANTUM GROUPS

Let $q^N = 1$, an odd root of unity, $\mathfrak{g}$ a semisimple Lie algebra.[2]

$$\mathrm{u}_q(\mathfrak{g}) = \mathrm{u}_q(\mathfrak{n}_-) \otimes \mathrm{u}_q(\mathfrak{t}) \otimes \mathrm{u}_q(\mathfrak{n}_+)$$

$\mathfrak{g}$ — semisimple Lie algebra $\mathfrak{n}_\pm$ — nilpotent parts $\mathfrak{t}$ — Cartan part

Theorem (Drinfeld/Lusztig)

The quantum group $\mathrm{u}_q(\mathfrak{g})$ is a quotient of the Drinfeld double $\mathrm{Drin}(\mathrm{u}_q(\mathfrak{b}_-))$ of its Borel part $\mathrm{u}_q(\mathfrak{b}_-)$.

$\mathrm{Drin}(\mathrm{u}_q(\mathfrak{b}_-))$ is defined on $\mathrm{u}_q(\mathfrak{n}_-) \otimes \mathrm{u}_q(\mathfrak{t}) \otimes \mathrm{u}_q(\mathfrak{t})^* \otimes \mathrm{u}_q(\mathfrak{n}_+)$

$$\Longrightarrow \mathcal{Z}(\mathrm{u}_q(\mathfrak{b}_-)\text{–}\mathbf{mod}) \simeq \mathrm{Drin}(\mathrm{u}_q(\mathfrak{b}_-))\text{–}\mathbf{mod} \text{ is } \textit{too large}$$

Alternative: *Use a relative version $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ of the monoidal center.*

[2] N coprime to 3 if G_2-type

Braided Hopf Algebras

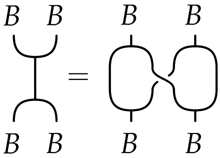
Idea: Take the Drinfeld double of $u_q(\mathfrak{n}_-) \subseteq u_q(\mathfrak{b}_-)$

Problem: $u_q(\mathfrak{n}_-)$ is *not* a Hopf algebra in $\mathbf{vect}_\Bbbk$
Solution: $u_q(\mathfrak{n}_-)$ is a *braided* Hopf algebra in $\mathbf{vect}_\Bbbk^\Lambda$

$\mathbf{vect}_\Bbbk^\Lambda$: Λ-graded vector spaces ($\Lambda = (\mathbb{Z}_N)^{\mathrm{rank}\,\mathfrak{g}}$) with braiding

$$\Psi_{V,W}(v \otimes w) = q^{\deg(v)\deg(w)} w \otimes v, \qquad q^N = 1$$

Bialgebra condition involves braiding



THE RELATIVE MONOIDAL CENTER

Assumption: $\mathcal{C}$ is *$\mathcal{B}$-central*, i.e. comes with a tensor functor

$$\mathcal{B}^{\mathrm{rev}} \hookrightarrow \mathcal{Z}(\mathcal{C})$$

that is faithful and preserves the braiding $\Psi^{\mathrm{rev}}_{V,W} = \Psi^{-1}_{W,V}$.

Note: Using the forgetful tensor functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, a $\mathcal{B}$-central structure gives a tensor functor $\mathcal{B} \to \mathcal{C}$ together with $\otimes$-compatible isomorphisms

$$\sigma_{V,B} \colon V \otimes B \xrightarrow{\sim} B \otimes V, \qquad \forall B \in \mathcal{B}, V \in \mathcal{C}.$$

Definition

$\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ is the full subcategory of $\mathcal{Z}(\mathcal{C})$ on objects (V, c) where

$$c_{V,B} = \sigma_{V,B}, \quad \forall B \in \mathcal{B}, V \in \mathcal{C}.$$

Quantum Groups Example

Proposition (Majid, L.)

Consider

$$\mathcal{B} = \mathbf{vect}_{\Bbbk}^{\Lambda}, \qquad \mathcal{C} = \mathrm{u}_q(\mathfrak{n}_-)\text{–}\mathbf{mod}(\mathcal{B}),$$
$$\mathcal{B}^{\mathrm{rev}} \hookrightarrow \mathcal{C}, \quad B \mapsto B \quad \textit{with trivial } u_q(\mathfrak{n}_-)\textit{-action via } \varepsilon.$$

Then $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ *is equivalent to* $\mathrm{u}_q(\mathfrak{g})$–$\mathbf{mod}$.

More generally: B a braided Hopf algebra in H–$\mathbf{mod}$,
$\mathcal{B} = H$–$\mathbf{mod}$ and $\mathcal{C} = B$–$\mathbf{mod}(\mathcal{B})$ then

$$\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \simeq \mathrm{Drin}_H(B)\text{–}\mathbf{mod},$$

a *braided* version of the Drinfeld double defined on $B^* \otimes H \otimes B$ [Majid's *double bosonization*], satisfying $\mathrm{Drin}_{\Bbbk}(B \rtimes H) \twoheadrightarrow \mathrm{Drin}_H(B)$.

MODULARITY OF THE CENTER

Let $\mathcal{C}$ be a finite tensor category. There is a *distinguished invertible object* D such that $D \otimes D^* \cong \mathbb{1}$ and a natural isomorphism

$$\xi_V \colon D \otimes V \xrightarrow{\sim} V^{****} \otimes D$$

called the *Radford isomorphism.*

Theorem (Etingof–Nikshych–Ostrik, Shimizu [Shi18])

(i) *The center $\mathcal{Z}(\mathcal{C})$ is non-degenerate finite tensor category.*

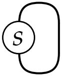
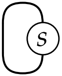
(ii) *Ribbon structures on $\mathcal{Z}(\mathcal{C})$ are in bijection with the set*

$$\mathrm{Sqrt}(D,\xi) = \{(V,\sigma_V)\,|\, V^{**} \otimes V \cong D,\ \sigma_V^{**}\sigma_V \cong \xi_V\}$$

Hence $\mathrm{Sqrt}(D,\xi) \neq \varnothing \Longrightarrow \mathcal{Z}(\mathcal{C})$ *is a modular tensor category.*

Modularity of the center

- In the *semisimple* case Müger (2001) proved that $\mathcal{Z}(\mathcal{C})$ is modular fusion provided $\mathcal{C}$ is *(trace) spherical*, i.e. for any $s\colon X \to X$

$\mathrm{tr}^l(s) =$  $=$  $= \mathrm{tr}^r(s).$

- Kauffman–Radford (1991) parametrized ribbon structures on $\mathrm{Drin}(H)$–**mod** for a fin. dim. Hopf algebra H by the set

$$\left\{ (\zeta, a) \in G(H^*) \times G(H) \,\middle|\, \zeta^2 = \alpha_H, a^2 = g_H \right\},$$

$g_H \in G(H)$ and $\alpha_{H^*} \in G(H^*)$ are the *distinguished grouplikes*.

In this case, $D = \Bbbk\langle d \rangle$, with $h \cdot d = \alpha_H^{-1}(h)d$, and ξ corresponds to action with g_H, using *Radford's formula* (1976)

$$S^4(h) = \alpha_H^{-1}(h_{(1)}) g_H h_{(2)} g_H^{-1} \alpha_H(h_{(3)}), \qquad \Delta(h) = h_{(1)} \otimes h_{(2)}.$$

Modularity of relative centers

Theorem (Shimizu [Shi19] $\mathcal{B} = \mathbf{vect}_{\Bbbk}$, L.–Walton [LW21])

Let $\mathcal{B}$ be a non-degenerate braided tensor category, $\mathcal{C}$ a $\mathcal{B}$-central tensor category such that the full image of $\mathcal{B}$ is a topologizing subcategory satisfy $\mathrm{Sqrt}(D, \xi) \neq \varnothing$.
Then $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ is a modular tensor category.

- A *topologizing subcategory* is a full subcategory closed under subquotients.
- The condition for ribbon structures is the same as for $\mathcal{Z}(\mathcal{C})$. The relative center inherits its ribbon structure.
- For non-degeneracy we prove and apply a more general result: For the *Müger centralizer* of a topologizing tensor subcategory $\mathcal{S}$,
$$\mathrm{Cent}_{\mathcal{C}}(\mathcal{S}) = \{X \in \mathcal{C} \mid \Psi_{V,X}\Psi_{X,V} = \mathrm{Id}_{X\otimes V}, \forall V \in \mathcal{S}\}$$
If $\mathcal{C}$ and $\mathcal{S}$ are non-degenerate then so is $\mathrm{Cent}_{\mathcal{C}}(\mathcal{S})$.
- The relative center is the Müger centralizer $\mathrm{Cent}_{\mathcal{Z}(\mathcal{C})}(\mathrm{Img}\,\mathcal{B}^{\mathrm{rev}})$.

Corollaries

Corollary (L.–Walton [LW21])

Under the assumptions of the theorem, there is a braided equivalence of ribbon categories

$$\mathcal{Z}(\mathcal{C}) \simeq \mathcal{B}^{\mathrm{rev}} \boxtimes \mathcal{Z}_{\mathcal{B}}(\mathcal{C}).$$

The following special case is of particular interest:

Definition (Douglas–Schommer-Pries–Snyder, 2013)

A tensor category with a pivotal structure $j_X \colon X \xrightarrow{\sim} X^{**}$ is *spherical* if $D \cong \mathbb{1}$ and $j_X^{**} j_X = \xi_X$.

Corollary (Shimizu [Shi19] $\mathcal{B} = \mathbf{vect}_{\Bbbk}$, L.–Walton [LW21])

Let $\mathcal{C}$ be a $\mathcal{B}$-central spherical tensor category such that the full image of $\mathcal{B}$ is a topologizing subcategory.
If $\mathcal{B}$ is non-degenerate, then $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ is modular.

Nichols algebra examples

Rich class of braided Hopf algebras, Nichols algebras:
braided Hopf algebras determined by pairs (V, c)

- $V = \mathbb{C}\langle x_1, \ldots, x_r\rangle$ a f.d. $\mathbb{C}$-vector space
- $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \qquad q_{ij} \in \mathbb{C}^\times$ roots of unity.
- The scalars $\mathbf{q} = (q_{ij})$ determine a braiding on $\mathcal{B}_\mathbf{q} = \mathbf{vect}_\Bbbk^\Lambda$ for a finitely generated abelian group $\Lambda = \langle g_1, \ldots, g_r\rangle$.

The tensor algebra $T_\mathbf{q}(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ becomes an (infinite-dimensional) Hopf algebra in $\mathcal{B}_\mathbf{q}$ by uniquely extending

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$$

to give a coproduct. E.g.,

$$\Delta(x_i x_j) = x_i x_j \otimes 1 + x_i \otimes x_j + q_{ij} x_j \otimes x_i + 1 \otimes x_i x_j,$$

$\deg(x_i) = g_i, \deg(x_j) = g_j$.

Nichols algebra examples

Definition

The *Nichols algebra* $\mathfrak{B}_{\mathbf{q}}(V)$ is the maximal quotient of $T_{\mathbf{q}}(V)$ that is still a $\mathbb{Z}$-graded Hopf algebra in $\mathbf{vect}_{\Bbbk}^{\Lambda}$ by an ideal generated in degrees ≥ 2.

Example: $u_q(\mathfrak{n}_-)$ is a Nichols algebra, for $q_{ij} = q^{a_{ij}}$

- The Nichols algebras studied here are called *diagonal type* and finite-dimensional examples have been classified completely in terms of *generalized Dynkin diagrams*

no edge if $q_{ij}q_{ji} = 1$

[Heckenberger, 2009]

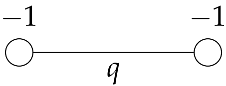
- Vast supply of examples generalizing $u_q(\mathfrak{n}_-)$ — *super type*, *modular type*, *UFO type*,... governed by a Weyl group*oid* action

NICHOLS ALGEBRA EXAMPLES

- We identified numerical conditions on the scalars $\mathbf{q} = (q_{ij})$ under which $\mathrm{Drin}_{\mathbb{C}[\Lambda]}(\mathfrak{B}(V))$–**mod** is a *modular tensor category*.
- Similarly, under certain conditions $\mathfrak{B}(V) \rtimes \mathbb{C}[\Lambda]$–**mod** is a non-semisimple *spherical category*.
- As a special case, $\mathrm{u}_q(\mathfrak{g})$–**mod** is a modular category (orginally due to Lyubashenko, 1995).
- However, $\mathrm{u}_q(\mathfrak{b}_-)$–**mod** is *not* non-semisimple spherical.
- Next slide: An example of super type that gives a non-semisimple spherical structure.

Examples beyond $\mathbf{u}_q(\mathfrak{g})$

Consider the generalized Dynkin diagram of super type $A(1|1)$:



, $\quad q$ — a primitive $2n$-th root of unity.

The Hopf algebra $U_2 := \mathrm{Drin}_{\mathbb{C}[\Lambda]}(\mathfrak{B}(V))$ is generated by x_i, y_i, and k_i (for $i = 1, 2$) subject to relations, for $i, j = 1, 2$, $i \neq j$,

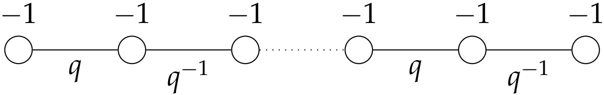
$$
\begin{gathered}
k_i x_i = q x_i k_i, \qquad k_i y_i = q^{-1} y_i k_i, \qquad k_i x_j = x_j k_i, \qquad k_i y_j = y_j k_i, \\
y_i x_i + x_i y_i = \delta_{i,j}(1 - k_i), \qquad y_1 x_2 = x_2 y_1, \qquad y_2 x_1 = q x_1 y_2, \\
x_i^2 = y_i^2 = 0, \quad k_i^{2n} = 1, \quad (x_1 x_2)^{2n} + (x_2 x_1)^{2n} = (y_1 y_2)^{2n} + (y_2 y_1)^{2n} = 0, \\
\Delta(x_1) = x_1 \otimes 1 + k_1^n \otimes x_1, \qquad \Delta(x_2) = x_2 \otimes 1 + k_2^n k_1 \otimes x_2, \\
\Delta(y_1) = y_1 \otimes 1 + k_1^n k_2 \otimes y_1, \qquad \Delta(y_2) = y_2 \otimes 1 + k_2^n \otimes y_2.
\end{gathered}
$$

EXAMPLES BEYOND $\mathbf{u}_q(\mathfrak{g})$

Proposition (L.–Walton)

*The category $\mathfrak{B}(V) \rtimes \mathbb{C}[\Lambda]$–**mod** is non-semisimple spherical in the previous example. Hence, U_2–**mod** is a modular tensor category.*

- Joint work in progress with G. Sanmarco: U_2 is part of a family U_{2r} of Hopf algebras such that U_{2r}–**mod** is *modular* based on a *spherical structure* of the bosonization of the Nichols algebra — of super type $A(r|r)$.



- There 2^r other *ribbon structures* on $\mathrm{Drin}(\mathfrak{B}(V) \rtimes \mathbb{C}[\Lambda])$ *not* coming from spherical structures, descending to the same one on the quotient U_{2r}.
- These are the *only** super A type examples of Nichols algebras admitting spherical structures after bosonization.

Local modules over braided commutative algebras

- $\mathcal{C}$ — a finite ribbon category (braiding Ψ)
- A — a *commutative algebra* in $\mathcal{C}$ — $m\Psi_{A,A} = m$
- $\mathrm{Rep}_{\mathcal{C}}(A) = (A\text{–}\mathbf{mod}, \otimes_A)$ is a finite tensor category

Theorem (Pareigis)

A module $(V, a_V \colon A \otimes V \to V)$ in $\mathrm{Rep}_{\mathcal{C}}(A)$ is local if

$$a_V c_{A,V} = a_V c_{A,V}^{-1}.$$

The full subcategory $\mathrm{Rep}_{\mathcal{C}}^{\mathrm{loc}}(A)$ of local modules is braided with braiding $\Psi_{V,W} \colon V \otimes_A W \to W \otimes_A V$ induced from $\mathcal{C}$.

MODULARITY OF LOCAL MODULES

Theorem (Kirillov–Ostrik [KO02], L.–Walton)

Let $\mathcal{C}$ be a modular category and A a rigid Frobenius algebra in $\mathcal{C}$. Then $\mathrm{Rep}^{\mathrm{loc}}_{\mathcal{C}}(A)$ is a modular category

- The case when $\mathcal{C}$ is *semisimple* is due to Kirillov–Ostrik
- We define a rigid Frobenius algebra A to be a *commutative, connected, special* **Frobenius algebra**
  - $\Leftrightarrow$ *connected étale algebra (i.e., commutative and separable)* A with $\dim_{\mathcal{C}} A \neq 0$ and trivial twist
  - Recovers the rigid $\mathcal{C}$-algebras of Kirillov–Ostrik
- **Examples:** Davydov's classification of connected étale algebras in $\mathcal{Z}(\Bbbk G\text{–}\mathbf{mod})$, G a finite group, extends to arbitrary characteristic
- Classification of rigid Frobenius algebras in $\mathcal{Z}(\mathbf{vect}_G^{\omega})$, joint with A. Ros Camacho, S. Hannah (Cardiff), forthcoming

LOCAL MODULES AND RELATIVE CENTERS

- $\mathcal{C}$ — $\mathcal{B}$-central finite tensor category
- (A, c_A) — an algebras in $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$
- Use the half-braiding c_A to make $\mathrm{Rep}_{\mathcal{C}}(A)$ a $\mathcal{B}$-central tensor category

Theorem (Schauenburg [Sch01] $\mathcal{B} = \mathbf{vect}_{\Bbbk}$, L.–Walton [LW20])

Assume that $\mathcal{B}$ is non-degenerate, $\mathrm{Sqrt}(D, \xi) \neq \varnothing$, and A is rigid Frobenius.
There is an equivalence of modular categories

$$\mathcal{Z}_{\mathcal{B}}(\mathrm{Rep}_{\mathcal{C}}(A)) \cong \mathrm{Rep}^{\mathrm{loc}}_{\mathcal{Z}_{\mathcal{B}}(\mathcal{C})}(A).$$

This result generalizes a theorem of Schauenburg [Sch01] to the relative setup.

Thank you very much for your attention!

SOME REFERENCES

[DRGG$^+$19] M. De Renzi, A. M Gainutdinov, N. Geer, B. Patureau-Mirand, and I. Runkel, *3-dimensional TQFTs from non-semisimple modular categories*, arXiv:1912.02063 (2019).

[KO02] A. Kirillov Jr. and V. Ostrik, *On a q-analogue of the McKay correspondence and the ADE classification of $\mathfrak{sl}_2$ conformal field theories*, Adv. Math. **171** (2002), no. 2, 183–227.

[LW20] R. Laugwitz and C. Walton, *Constructing non-semisimple modular categories with local modules*, arXiv:2202.08644 (2020).

[LW21] ______, *Constructing Non-Semisimple Modular Categories With Relative Monoidal Centers*, IMRM (2021). rnab097.

[Sch01] P. Schauenburg, *The monoidal center construction and bimodules*, J. Pure Appl. Algebra **158** (2001), no. 2-3, 325–346.

[Shi18] K. Shimizu, *Ribbon structures of the Drinfeld center of a finite tensor category*, arXiv:1707.09691 (2018).

[Shi19] ______, *Non-degeneracy conditions for braided finite tensor categories*, Adv. Math. **355** (2019), 106778, 36.

[SW21] C. Schweigert and L. Woike, *Homotopy coherent mapping class group actions and excision for Hochschild complexes of modular categories*, Adv. Math. **386** (2021), Paper No. 107814, 55.