2-*Representation Theory for Categorified Quantum Groups at Prime Roots of Unity*

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joint work with

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Outline

Motivation
from categorification
(at prime roots of 1)
Khovanov–Elias–Qi

$+$

2-Representation Theory
(Mazorchuk–Miemietz)

p-dg 2-Representations

CATEGORIFICATION

Vision (Crane–Frenkel, 1994):
invariants of 3-manifolds $\rightsquigarrow$ invariants of 4-manifolds

$$\begin{tikzcd}
\text{Algebras} \arrow[rr, bend right, "\textit{Categorification}"'] & & \text{Categories} \arrow[ll, bend right, "K_0(-)\text{ Grothendieck ring}"'] \\
& \begin{array}{c}\text{describe 3-manifold invariants}\\ \text{via representation theory}\end{array} \arrow[ul] &
\end{tikzcd}$$

$$\begin{tikzcd}
\text{2-category } \mathscr{C} \arrow[r, "\text{decategorify}"] & \text{algebra/category } K_0(\mathscr{C}) \\
\text{objects } \mathtt{i}, \mathtt{j}, \ldots \arrow[r, mapsto] & \text{idempotents/objects} \\
\text{1-morphisms } \mathtt{i} \xrightarrow{\mathrm{F}} \mathtt{j},\ \mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j}) \arrow[r, mapsto] & K_0(\mathscr{C}(\mathtt{i}, \mathtt{j})) \\
\text{2-Homs } \mathrm{Hom}_{\mathscr{C}}(\mathrm{F}, \mathrm{G}) \arrow[r, mapsto] & \text{relations}
\end{tikzcd}$$

CATEGORIFICATION OF QUANTUM GROUPS

Quantum groups:

- $\mathrm{U}_q(\mathfrak{g})$, q-deformation of $\mathrm{U}(\mathfrak{g})$ (Drinfeld, Jimbo)
- small quantum group: $\mathrm{u}_q(\mathfrak{g})$ finite-dimensional quotient, for q root of unity
- Categories $\dot{\mathrm{U}}_q(\mathfrak{g})$, $\dot{\mathrm{u}}_q(\mathfrak{g})$, objects are idempotents to ensure lattice grading (Lusztig)

Categorifications:

- at *generic q*: Khovanov–Lauda, Chuang–Rouquier, ...
- *Khovanov homology*, categorified Jones polynomial of links
- at *prime roots of unity*, $q^p = 1$:
  - Khovanov–Qi (2012), Elias–Qi (2013): small quantum groups $\dot{\mathrm{u}}_q(\mathfrak{sl}_2)$
  - Elias–Qi (2015): quantum group $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$

CATEGORIFICATION OF QUANTUM GROUP FOR $\mathfrak{sl}_2$ (LAUDA)

Categorification		Decategorification
Diagrammatic 2-category $\mathscr{U}$	$\longmapsto$	$\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$,
objects $\mathbb{1}_\lambda, \lambda \in \mathbb{Z}$	$\longmapsto$	idempotents 1_λ
indecomposable 1-morphisms $\mathrm{F}^{(a)}\mathrm{E}^{(b)}\mathbb{1}_\lambda, \mathrm{E}^{(a)}\mathrm{F}^{(b)}\mathbb{1}_\lambda$ $\lambda \leq b-a$	$\longmapsto$	canonical basis $F^{(a)}E^{(b)}1_\lambda, E^{(a)}F^{(b)}1_\lambda$ $\lambda \leq b-a$
2-Homs $\mathrm{End}_{\mathscr{U}}(\mathrm{E}^m\mathbb{1}_\lambda) \cong \mathrm{Sym} \otimes \mathrm{NH}_m$	$\longmapsto$	relations

indecomposable 1-morphisms $\xrightarrow{\text{realized as}}$ indecomposable bimodules over nil-Hecke algebras NH_m

Root at unity categorification

Crucial difference:

At q generic:

- $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$ is a $\mathbb{Z}[q, q^{-1}]$-algebra
- q-action $\longleftrightarrow$ grading shift functors

At prime root of 1:

- $\dot{\mathrm{u}}_q(\mathfrak{sl}_2)$ algebra over $\mathbb{Z}[q, q^{-1}]/(1 + q + q^2 + \ldots + q^{p-1})$
  cyclotomic integers
  $\longrightarrow$ categorified: stable category of (graded) H-mod
  (Khovanov, 2005)
- $H = \Bbbk[\partial]/(\partial^p)$ Hopf algebra in char $\Bbbk = p$
  $\longrightarrow$ (graded) algebras with p-differentials:
  p-dg algebras

CATEGORIFICATION OF QUANTUM GROUPS AT ROOTS OF UNITY

Khovanov 2005: **stable category** of H-mod, $H = \Bbbk[\partial]/(\partial^p)$ categorifies $\mathbb{Z}[q]/(1+q+\ldots+q^{p-1})$

$\rightsquigarrow$ ***Hopfological Algebra***, study of stable categories of p-dg modules over a p-dg algebra

Theorem (Elias–Qi, 2013)

The diagrammatic category $\mathcal{U} = \bigoplus_{\lambda,\mu\in\mathbb{Z}} \mathcal{U}^\lambda$ (Lauda) equipped with a p-differential categorifies the idempotent version of the small quantum group $\dot{\mathrm{u}}_q(\mathfrak{sl}_2)$, i.e.

$$\bigoplus_{\lambda\in\mathbb{Z}} K_0(\mathcal{D}_H(\mathcal{U}^\lambda\text{-mod})) \cong \dot{\mathrm{u}}_q(\mathfrak{sl}_2)$$

- $L(\lambda)$ is categorified by ***cyclotomic quotient 2-category***
- Elias–Qi (2015) also categorified $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$

Question: How to categorify representations?

$$A \xrightarrow{\Bbbk\text{-linear functor}} \mathrm{Vect}_{\Bbbk},$$

A is a $\Bbbk$-linear category with one object,
or objects $\longleftrightarrow$ idempotents.

Categorified:

$$\mathscr{C} \xrightarrow{\mathbf{M},\ \Bbbk\text{-linear 2-functor}} \mathfrak{R}_{\Bbbk},$$

2-category

suitable 2-category to represent in
Objects: A-mod, A finite-dim.
1-morphisms: projective bimodules
2-morphisms: bimodule homs

OVERVIEW

2-representations in the literature:

- Rouquier 2004, 2008 (2-Kac Moody algebras)
- E.g. Khovanov–Mazorchuk–Stroppel 2008 (Specht modules)
- Mazorchuk–Miemietz 2010–... (systematic study)
  - $\mathscr{C} \longrightarrow \mathfrak{R}_{\Bbbk}$ 2-functor
  - finiteness conditions $\mathscr{C}$:
    - finitely many objects
    - finitely many indecomposable 1-morphisms
    - finite-dimensional 2-Homs $\Rightarrow$ *finitary 2-category*
    - plus adjoints $\Rightarrow$ *fiat 2-categories*
  - target $\mathfrak{R}_{\Bbbk}$:
    - objects: small $\Bbbk$-Karoubian (or abelian) categories $\cong A$-mod
    - 1-morphisms $\longleftrightarrow$ tensoring with projective bimodules
    - 2-morphisms $\longleftrightarrow$ bimodule homs

SIMPLE TRANSITIVE 2-REPRESENTATIONS

Question: What are ***simple*** 2-representations?

For algebra representations:

$$\forall v, w \neq 0 \ \ \exists a \in A: \quad av = w \qquad\qquad \text{“transitivity”}$$

For a 2-representations $\mathbf{M}\colon \mathscr{C} \longrightarrow \mathfrak{R}_{\Bbbk}$:

(1) transitivity on 1-morphisms

(2) there is no non-trivial ideal in $\mathbf{M}$ closed under $\mathscr{C}$-action not containing identity 2-morphisms

then $\mathbf{M}$ is ***simple transitive*** (Mazorchuk–Miemietz, 2014)

Question: How to construct simple transitive 2-representations?

Answer: 2-Cell representations! (Mazorchuk–Miemietz, 2010)

(inspired by Kazhdan–Lusztig cells for Hecke algebras, cellular algebras)

Partial order on 1-Morphisms:

$$\mathrm{F} \leq_L \mathrm{G} \qquad \Longleftrightarrow \qquad \mathrm{G} \text{ direct summand } \mathrm{H} \circ \mathrm{F}, \text{ some } \mathrm{H}$$

Equivalence classes: ***left cells*** of $\mathscr{C} \rightsquigarrow \mathcal{L} \subset \mathscr{C}(\mathtt{i}, -)$

Cell 2-representations $\mathbf{C}_{\mathcal{L}}$:

- *Principal 2-representations* $\mathbf{P}_{\mathtt{i}} : \mathtt{j} \mapsto \mathscr{C}(\mathtt{i}, \mathtt{j}), \quad \mathrm{G} \mapsto \mathrm{G} \circ (-)$
- 2-subrep $\mathbf{R}_{\mathcal{L}} \leq \mathbf{P}_{\mathtt{i}}$ generated by 1-morphism in the cell $\mathcal{L}$
- Maximal quotient $\mathbf{C}_{\mathcal{L}} := \mathbf{R}_{\mathcal{L}} / \mathbf{I}$ not annihilating identities

Partial converse: Simple transitive $\Longrightarrow$ 2-cell rep (MM, 2014)

Example $\mathfrak{sl}_2$

$\mathscr{U}$ categorification of $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$ of Lauda:

- objects $\lambda \in \mathbb{Z}$ weights
- generating 1-morphisms $\mathbb{1}_\lambda$, $\mathrm{E}_\lambda^{(a)}\mathrm{F}_\lambda^{(b)}$, $\mathrm{F}_\lambda^{(a)}\mathrm{E}_\lambda^{(b)}$, $\lambda \leq b - a$
- 2-morphisms: string and bubble diagrams, equivalently, endomorphism of nilHecke algebra bimodules

Rouquier (2008), Webster (2013):
$\mathbf{L}_\lambda \colon \mathscr{U} \longrightarrow \mathfrak{R}_{\Bbbk}$, categorifies simple $L(\lambda)$ via ***cylotomic quiver Hecke (KLR) algebras*** $\bigoplus_{n\geq 0} R_n^\lambda$-proj

- Quotient $\mathscr{U}^\lambda := \mathscr{U}/\ker(\mathbf{L}_\lambda)$ is fiat, and the cell 2-representations are categorifications of $L(\lambda)$
- these are the only simple transitive 2-representations of $\mathscr{U}^\lambda$ (up to equivalence)
- "Schur Lemma": $\mathrm{End}(\mathbf{L}_\lambda) \cong \Bbbk$-mod (MM, 2015)
  (The last statement fails for type B)

p-DG 2-REPRESENTATION THEORY

Recent work with V. Miemietz (East Anglia)

- A theory of 2-Representation following Mazorchuk–Miemietz in a p-dg enriched setting
- Construction of cell 2-representations in this setup
- Derive to get 2-representations of triangulated, stable, 2-categories
- Applicable to the categorified representations of $\dot{\mathrm{u}}_q(\mathfrak{sl}_2)$ (cyclotomic quotients)

THE SETUP

p-dg enriched 2-category $\mathscr{C}$

- $\mathtt{i}$, $\mathtt{j}$,... finitely many objects
- $\mathscr{C}(\mathtt{i},\mathtt{j})$ small category enriched with p-differential

Suitable finiteness assumptions: strongly finitary p-dg categories $\mathscr{C}(\mathtt{i},\mathtt{j})$

- the $\Bbbk$-linear category is finitary and Karoubian
- all subquotients exist in the enriched category (not necessary closed under ∂)
- all objects are filtered by $\Bbbk$-indecomposables and cofibrant, (*fantastic filtration* of Elias–Qi)
  $\rightsquigarrow$ *strongly finitary p-dg category*

Example: $D = \Bbbk[x]/(x^p)$, $\mathbb{1}$, $\mathrm{F} = D \otimes_{\Bbbk} D$ generating 1-morphisms, $\partial(x) = x^2$ differential on $\mathrm{End}(\mathbb{1}) = D$, $\mathrm{End}(\mathrm{F}) = D \otimes D$

p-DG 2-REPRESENTATIONS

Question: What is the target? $\mathbf{M}\colon \mathscr{C} \longrightarrow ??$

Definition

A ***p*-dg 2-representation** is a p-dg 2-functor $\mathscr{C} \longrightarrow \mathfrak{M}_p$

Target $\mathfrak{M}_p$:

- objects: small p-dg categories $\overline{\mathcal{A}} \cong \mathcal{A}$-csf
  compact semi-free modules over a strongly finitary p-dg category
  (We employ a combinatorial description generalizing *twisted complexes* of Bondal–Kapranov)
- 1-morphisms: tensoring by cofibrant bimodules
- 2-morphisms: bimodule morphisms (enriched)

p-DG 2-REPRESENTATIONS

Technical aspects:

- p-dg modules are p-dg functors to $\Bbbk\text{-mod}_H$
- *Cofibrant modules* have nice filtration by representable objects.
- subquotient idempotent completion
- closure under all quotients, "enriched abelianization"
- Yoneda Lemma $\mathrm{Hom}(\mathbf{P}_{\mathtt{i}}, \mathbf{M}) \cong \mathbf{M}(\mathtt{i})$ equivalence of p-dg categories

Theorem (L.–Miemietz)

A p-dg 2-representation $\mathbf{M}\colon \mathscr{C} \to \mathfrak{M}_p$ *induces a triangulated 2-representation* $\mathbf{KM}$ *of the triangulated 2-category* $\mathscr{K}(\mathscr{C})$ *(obtained by taking stable categories* $\mathcal{K}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*).*

RESULTS

Assume: $\mathscr{C}$ strongly finitary p-dg 2-category

We can define:

- analogue of principal 2-representations $(\mathbf{P}_{\mathtt{i}}, \mathbf{C}_{\mathcal{L}})$
  $\mathbf{P}_{\mathtt{i}}(\mathtt{j}) = \overline{\mathscr{C}(\mathtt{i}, \mathtt{j})}$
- analogue of cell 2-representations $(\mathbf{P}_{\mathtt{i}}, \mathbf{C}_{\mathcal{L}})$
  first: restrict to cell, second: take a maximal quotient.

Theorem (L.–Miemietz)

For a strongly finitary 2-category $\mathscr{C}$, the cell 2-representations are simple transitive p-dg 2-representations.
The underlying additive 2-representations are inflations of the cell 2-representations of Mazorchuk–Miemietz by a local algebra.

Special class of examples:

The p-dg 2-categories $\mathscr{C}_{\mathcal{A}}$:
$\mathcal{A} = \prod_{i=1}^{n} \mathcal{A}_i$, a product of strongly finitary p-dg categories

- objects: $\mathtt{i} \longleftrightarrow \overline{\mathcal{A}}_i$
- 1-morphisms generated by: tensoring with cofibrant ($\Bbbk$-indecomposable) $\mathcal{A}_i \otimes \mathcal{A}_j^{\mathrm{op}}$-bimodules
- 2-morphisms: Morphisms of bimodules

Theorem (L.–Miemietz)

Assume $\partial(\mathrm{rad}\mathcal{A}) \subset \mathrm{rad}\mathcal{A}$, $\mathcal{A}$ strongly finitary.
Then $\mathbf{C}_{\mathcal{L}} \cong \mathbf{N}$, where $\mathbf{N}$ is the natural (defining) 2-representation
($\mathcal{L}$ is the unique non-identity cell)

CATEGORIFICATION OF SMALL QUANTUM GROUPS

Via cyclotomic quotients $\mathbf{L}_\lambda \colon \mathscr{U} \longrightarrow \mathfrak{R}_{\Bbbk}$ categorifies simple $L(\lambda)$ for small quantum group $\dot{u}_q(\mathfrak{sl}_2)$ [Elias–Qi]
$\longrightarrow \mathscr{U}^\lambda := \mathscr{U}/\ker \mathbf{L}_\lambda$ is strongly finitary
Divided powers $\mathrm{F}^{(r)}\mathbb{1}_\lambda\mathrm{E}^{(s)} \longleftrightarrow$ tensoring by $NH_r^\lambda e_r \otimes e_s^* NH_s^\lambda$
generate lowest cell $\mathcal{L}$ w.r.t $\leq_L$

$$\mathrm{End}_{\mathscr{V}^\lambda}(\mathrm{F}^{(r)}\mathbb{1}_\lambda) \cong H_r^\lambda \qquad \text{(coinvariant algebra)}$$

APPLICATIONS OF RESULTS

Corollary (L.–Miemietz)

$\mathscr{U}^{\lambda}_{\mathcal{L}} \approx \mathscr{C}_{\mathcal{A}}$ are p-dg biequivalent,
where $\mathcal{A}$ is generated by regular p-dg bimodules over
$NH^{\lambda} = \prod_{r=0}^{p-1} NH^{\lambda}_r$

Idempotent completion: $\widehat{\mathscr{U}^{\lambda}}_{\mathcal{L}} \approx \mathscr{C}_{\widehat{\mathcal{A}}} \approx \mathscr{C}_B$

B is the product of coinvariant algebras $B = \prod_{r=0}^{p-1} H^{\lambda}_r$

$\Longrightarrow$ *The 2-cell representation $\mathbf{C}_{\mathcal{L}}$ of $\widehat{\mathscr{U}^{\lambda}}$ categorifies $L(\lambda)$*

Theorem (L.–Miemietz)

Every endofunctor of the categorified simple representation $\mathbf{L}_{\lambda}$ of $\mathscr{U}^{\lambda}$ is p-dg equivalent to a fantastic filtration of the identity functor.

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