The Koszul property for algebras of quasi-Plücker coordinates

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CONTENTS

Summary: We define a quadratic-linear algebra of *quasi-Plücker coordinates* and show it is Koszul

PLÜCKER COORDINATES

Plücker coordinates describe the embedding

$$\begin{tikzcd} & G_{k,n} \arrow[r, hook] & \mathbb{P}^{\binom{n}{k}-1} & \\ \text{Grassmannian} \arrow[ur, bend left] & A \arrow[r, mapsto] & (p_I(A))_I & \text{Projective Space} \arrow[ul, bend right] \end{tikzcd}$$

Case $(k, n) = (2, 4)$ [Plücker 1865]

$$\text{Point in } G_{k,n} \longleftrightarrow k \times n\text{-matrix } A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}$$

$$I \subseteq \{1, \dots, n\} \longleftrightarrow p_I(A) := \begin{vmatrix} a_{1i_1} & \dots & a_{1i_k} \\ \vdots & \vdots & \vdots \\ a_{ki_1} & \dots & a_{ki_k} \end{vmatrix} \in \mathbb{C}[a_{ij} \mid i, j\,]$$

THE COORDINATE RING OF $G_{k,n}$

Coordinate rings: Quotient ring

$$\mathbb{C}[p_I] \twoheadrightarrow \mathcal{O}_{k,n} := \mathbb{C}[p_I]/\mathrm{K}$$

- $I \subseteq \{1, \ldots, n\}$ all subsets of size $|I| = k$
- p_I are the *Plücker coordinate functions*
- K is the ideal of *Plücker relations* [Weitzenbröck, 1923]

PLÜCKER RELATIONS

Relations in K:

- Permuting indices ⟶ skew-symmetry
- Plücker relations

$$\sum_{t=1}^{k+1}(-1)^t p_{I|j_t} p_{J\setminus j_t} = 0,$$

for $I = \{i_1, \ldots, i_{k-1}\}, J = \{j_1, \ldots, j_{k+1}\} \subseteq \{1, \ldots, n\}$.

$$\begin{aligned} \mathcal{O}_{2,4}: &\qquad & p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} &= 0, \\ \mathcal{O}_{3,6}: &\qquad & p_{123}p_{456} - p_{124}p_{356} + p_{125}p_{346} - p_{126}p_{345} &= 0, \\ && p_{123}p_{245} - p_{124}p_{235} + p_{125}p_{234} &= 0, \end{aligned}$$

and relations from permuting the indices

HOMOLOGICAL PROPERTIES

Theorem (Doubilet–Rota–Stein '74, Sturmfels–White '89)

The ring $\mathcal{O}_{k,n}$ *is G-quadratic, i.e.* K *has a quadratic Gröbner basis. In particular,* $\mathcal{O}_{k,n}$ *is Koszul. That is,*

$$\mathrm{Ext}_A^*(\mathbb{C},\mathbb{C}) = \bigoplus_i \mathrm{Ext}_A^{i,i}(\mathbb{C},\mathbb{C}) = A^! = \mathbb{C}[p_I^*]/(\mathrm{K}^\perp),$$

where $\mathrm{K}^\perp$ *is the orthogonal complement of the relations.*

QUASI-DETERMINANTS

Commutative case: Determinants of minors describe Plücker embedding
Noncommutative case: Replace them by an analogue of determinants in noncommutative variables — **Quasi-determinants** [Gelfand–Retakh 1991]

Example
$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ entries in a division ring — *four* quasi-determinants:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{11} = a_{11} - a_{12}a_{22}^{-1}a_{21}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{12} = a_{12} - a_{11}a_{21}^{-1}a_{22}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{21} = a_{21} - a_{22}a_{12}^{-1}a_{11}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{22} = a_{22} - a_{21}a_{11}^{-1}a_{12}$$

PROPERTIES OF QUASI-DETERMINANTS

- Analogues of *quotients* of determinants $(-1)^{i+j}|A|/|A^{ij}|$.
- Quasi-determinants may not exists
- Version of *Cramer's Rule*
- Well-behaved with *Gaussian Elimination*
- Satisfy a noncommutative *Sylvester identity* (*heredity priciple*, well-behaved with block decompositions)
- No easy product rule
- Application: *Noncommutative symmetric functions*

QUASI-PLÜCKER COORDINATES

A — generic $k \times n$-matrix with noncommuting variables

$$q_{ij}^{I} = q_{ij}^{I}(A) := \begin{vmatrix} a_{1i} & a_{1i_1} & \dots & a_{1i_{k-1}} \\ \vdots & & \vdots & \\ a_{ki} & a_{ki_1} & \dots & a_{ki_{k-1}} \end{vmatrix}_{1i}^{-1} \begin{vmatrix} a_{1j} & a_{1i_1} & \dots & a_{1i_{k-1}} \\ \vdots & & \vdots & \\ a_{kj} & a_{ki_1} & \dots & a_{ki_{k-1}} \end{vmatrix}_{1j}$$

$I = \{i_1, \dots, i_{k-1}\} \subseteq \{1, \dots, n\}$, with $i \notin I$

- vanish if $j \in I$, and $q_{ii}^{I} = 1$
- independent of order of I
- GL_n-invariant in A
- $q_{ij}^{N\setminus\{i,j\}} q_{jm}^{N\setminus\{j,m\}} = -q_{im}^{N\setminus\{i,m\}}$ *(Noncom. Skew-Symmetry)*
- If $i \notin M$, then $\sum_{j\in L} q_{ij}^{M} q_{ji}^{L\setminus\{j\}} = 1$ *(Plücker Relations)*

ALGEBRAS OF QUASI-PLÜCKER COORDINATES

$Q_n^{(k)}$ — algebra of quasi-Plücker coordinates, generated by q_{ij}^I, with $|I| = k-1$

$R_n^{(k)} \subseteq Q_n^{(k)}$ — subalgebra generated by q_{ij}^I, with $i < j$

Note 1: The assignment $q_{ij}^I \longmapsto q_{ij}^I(A)$
gives an **algebra homomorphism** from $Q_n^{(k)}$ to the free skew-field generated by the entries of A.
$\longrightarrow$ Both $R_n^{(k)}, Q_n^{(k)}$ generate the same sub skew-field

Note 2: If the entries of A commute, then

$$\underbrace{q_{ij}^I(A)}_{\text{quasi-Plücker}} = \underbrace{\frac{p_{j|I}(A)}{p_{i|I}(A)}}_{\text{Plücker}}$$

Proposition (L.–Retakh)

The algebra $R_n^{(k)}$ is a quadratic-linear algebra with generators q_{ij}^I, for $i < j$, such that

$$q_{ij}^I q_{jl}^I = q_{il}^I$$
$$\sum_{j=1}^{k-1} q_{l_0 l_j}^M q_{l_j l_k}^{L \setminus \{l_j, l_k\}} + q_{l_0 l_k}^{L \setminus \{l_0, l_k\}} = q_{l_0 l_k}^M$$

Example

$R_n^{(2)}$: $(n-2)\binom{n}{2}$ generators q_{ij}^k for $k \notin \{i < j\}$, such that

$$q_{ij}^m q_{jk}^m = q_{ik}^m, \qquad q_{ij}^m q_{jk}^i + q_{ik}^j = q_{ik}^m, \qquad m \notin \{i < j < k\}.$$

NONHOMOGENEOUS KOSZUL ALGEBRAS

Priddy 1970: *Koszul algebras* — quadratic graded algebras such that $\mathrm{Ext}_A^*(\Bbbk, \Bbbk)$ is concentrated in diagonal bi-degree (i.e. easy to compute).
Positselski 1993: Theory of *nonhomogeneous* Koszul algebras

Basic idea: Check Koszulity of the associated graded algebra.

- If A is quadratic-linear, then the Koszul dual $\mathrm{Ext}_A^*(\Bbbk, \Bbbk)$ becomes a *DG algebra*, otherwise a *curved DG algebra*.
- **Example:** $U(\mathfrak{g})$ universal enveloping algebra of a Lie algebra is nonhomogeneous Koszul.
  $\Longrightarrow$ If $\mathfrak{g}$ is semisimple, the dual is the standard Lie algebra cohomology complex.

THE MAIN THEOREM

Theorem (L.–Retakh)

The algebra $R_n^{(k)}$ *is a quadratic-linear Koszul algebra (in the sense of Positselski).*
Similarly, $Q_n^{(k)}$ *is a nonhomogeneous Koszul algebras.*

Proof strategy.

The quadratic parts of the relations in $R_n^{(k)}$ form a quadratic Gröbner basis (that is, have a noncommutative PBW basis). This can be shown using the quadratic dual (which is finite-dimensional). □

LINKS TO OTHER WORK

- Lauve 2005: Quasi-Plücker relations determine relations of the *q-Grassmannian* of Taft–Towber
- Sottile–Sturmfels 1999: coordinate rings of *Quantum Grassmannian* (i.e minors with polynomial entries) are Koszul
- The algebra $Q_n^{(2)}$ appears in Berenstein–Retakh's *Noncommutative Marked Surfaces*
- Similar relations to $Q_n^{(k)}$ appear in Pendavingh's study of *Matroids over Skew-Fields*

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